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Mathematical Reviews

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MATHEMATICAL REVIEWS

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Journal references in Mathematical Reviews are now given in the following form: J. Brobdingnag. Acad. Sci. (7) **4 (82)** (1952/53), no. 3, 17–42 (1954), where after the abbreviated title one has: (series number) volume number (volume number in first series if given) (nominal date), issue number if necessary, first page–last page (imprint date). In case only one date is given, this will usually be interpreted as the nominal date and printed immediately after the volume number (this is a change from past practice in Mathematical Reviews where a single date has been interpreted as the imprint date). If no volume number is given, the year will be used in its place. The symbol ★ precedes the title of a book or other non-periodical which is being reviewed as a whole.

References to reviews in Mathematical Reviews before volume 20 (1959) are by volume and page number, as MR 19, 532: from volume 20 on, by volume and review number, as MR **20** #4387. Reviews reprinted from Applied Mechanics Reviews, Referativnyĭ Žurnal, or Zentralblatt für Mathematik are identified in parentheses following the reviewer's name by AMR, RŽMat (or RŽMeh, RŽAstr. Geod.), Zbl, respectively.

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Mathematical Reviews

Vol. 22, No. 3A March, 1961 Reviews 1487–1927

GENERAL

See also 1513, B2030.

1487:

Good, I. J. A classification of fallacious arguments and interpretations. Methodos **11** (1959), 147–159.

A list (163 headings and subheadings), necessarily subjective, covering mathematics, statistics, and general discourse. It does not include typographical errors, of which an instance appears in the first subheading.

1488:

Maxwell, E. A. ★Advanced algebra. Part I. Cambridge University Press, New York, 1960. xii+311 pp. $2.75.

A text-book with standard material for secondary schools in Great Britain.

1489:

Sakovič, G. N. List of literature on mathematics published in the Ukraine in 1957. Ukrain. Mat. Ž. **11** (1959), 115–117. (Russian)

A list of the titles and tables of contents of eight publications.

HISTORY AND BIOGRAPHY

1490:

Vogel, Kurt. ★Vorgriechische Mathematik. II. Die Mathematik der Babylonier. Mathematische Studienhefte, 2. Hermann Schroedel Verlag KG, Hannover; Verlag Ferdinand Schöningh, Paderborn; 1959. 94 pp. Kart.: DM 12.00.

This second part of the author's *Vorgriechische Mathematik* [I: Schroedel, Hannover, 1958; MR **20** #5710] begins with a tabular survey of Babylonian chronological history. This is followed by a chapter giving a brief history of the lands bordering the lower Euphrates and Tigris rivers, the development of writing, a discussion of the table and problem texts, and a consideration of Sumerian or Babylonian origin of mathematics with the conclusion of Sumerian origin [K. Vogel, Math. Nachr. **18** (1958), 377–382; MR **21** #2562]. The book continues with information on the Babylonian system of numbers and measures, including the sexagesimal system. Under computation techniques are described the four usual operations and note is made of some interesting tables and texts dealing with exponents, square and cube roots, n^2+n^3, and $n^2(n-1)$. Among the arithmetic problems occur progressions, Pythagorean triples, and examples from daily life. The part on algebra treats of first degree equations, quadratic equations, cubic equations, and equations of higher degree, all with many examples. The geometrical problems, with many illustrative examples, are concerned with both plane and solid figures. There is a section on the nature of Babylonian mathematics. Finally, there is a short list of exercises, with some answers, and a bibliography. The book is illustrated with many photographs and line drawings related to the material discussed.

A useful addition to the bibliography is O. Neugebauer, *The exact sciences in antiquity* [2nd ed., Brown Univ., Providence, R.I., 1957; MR **19**, 825].

E. B. Allen (Troy, N.Y.)

1491:

Seidenberg, A. Peg and cord in ancient Greek geometry. Scripta Math. **24** (1959), 107–122.

The author expresses the view that the whole of Greek geometry had a ritual origin. No direct documentary evidence is given for this statement except for references to (1) the altar constructions in the *Sulvasutras*, an ancient work from India, and (2) an Egyptian ceremony for the old Kingdom. In the opinion of the author the practice of using peg and cord for constructions was diffused from a single center (no location is given) and its widespread occurrence is evidence for its antiquity. The date 2500 B.C. is mentioned.

Part of the paper contains an argument against Zeuthen's reconstructions of the early history of the conics. Contrary to Zeuthen the author maintains that the earliest definition of the ellipse was based upon the properties of the foci "because in their constructions with peg and cord the ritualists actually drew ellipses". It is further shown how a neusis construction (usually attributed to Archimedes) leading to the trisection of an angle can be performed by cord and peg, and from this observation it is inferred that the cord and peg constructions gave rise to a tradition of neusis constructions.

{The formula in the parenthesis on p. 116, l. 14–15, should read: $ax=cy+a^2-c^2$.} *O. Schmidt* (Copenhagen)

1492:

Miller, Maximilian. Aristarch von Samos als Vorläufer des Kopernikus: Aristarchs Schrift über die Grösse und Entfernung von Sonne und Mond. Wiss. Z. Hochsch. Verkehrswes. Dresden **6** (1958/59), 257–265. (Russian, English and French summaries)

Zusammenfassung des Verfassers: "In der Einleitung werden die aus dem Altertum überlieferten Mitteilungen über das Leben Aristarchs aufgeführt. Hieran schliesst

sich eine Übersetzung der noch erhaltenen astronomischen Schrift und der wichtigsten Fragmente Aristarchs. Zum Schluss wird die Frage erörtert, ob die Lehre Aristarchs die neuzeitliche Astronomie (Kopernikus) wesentlich beeinflusste." *L. Gyarmathi* (Debrecen)

1493:
Sayili, Aydin. Thâbit ibn Qurra's generalization of the Pythagorean theorem. Isis **51** (1960), 35–37.

1494:
Clavelin, Maurice. Le problème du continu et les paradoxes de l'infini chez Galilée. Thalès **10** (1959), 1–26 (1960).

1495:
Miller, Maximilian. Newton, Aufzählung der Linien dritter Ordnung. Wiss. Z. Hochsch. Verkehrswes. Dresden **1** (1953), no. 1, 5–32.

Der erste Teil der Arbeit enthält eine deutschsprachige Erörterung des Newtonschen Werkes "Enumeratio linearum tertii ordinis"; im zweiten Teil folgen Bemerkungen, Ergänzungen und Berichtigungen. Dabei berücksichtigt der Verfasser auch eine Schrift von James Stirling [*Lineae tertii ordinis Newtonianae*, 1717], welche eine Besprechung der genannten Arbeit von Newton darstellt. *L. Gyarmathi* (Debrecen)

1496:
Miller, Maximilian. Isaac Newton: Über die Analysis mit Hilfe unendlicher Reihen. Wiss. Z. Hochsch. Verkehrswes. Dresden **2** (1954), no. 2, 1–16.

In der vorliegenden Arbeit teilt der Verfasser die Newtonsche Arbeit "Analysis per aequationes numero terminorum infinitas" in deutscher Übersetzung und mit Bemerkungen versehen mit. Die Bemerkungen beziehen sich teilweise auf die Entstehungsgeschichte der Infinitesimalrechnung, teilweise aber geben sie eine modernere Interpretation bzw. Berichtigungen des Newtonschen Werkes. *L. Gyarmathi* (Debrecen)

1497:
Miller, Maximilian. Newtons Differenzenmethode. Wiss. Z. Hochsch. Verkehrswes. Dresden **2** (1954), no. 3, 1–13. (Russian, English and French summaries)

Im ersten Teil seiner Arbeit erörtert der Verfasser den Inhalt des Newtonschen Werkes "Methodus differentialis", wobei er die von Newton oft nur in skizzenhafter Weise angegebenen Beweise der Interpolationsformeln vervollständigt und ein sich als Folgerung ergebendes Kurvenquadraturverfahren mitteilt. Am Ende der Arbeit finden wir eine deutsche Übersetzung des angeführten Werkes von Newton. *L. Gyarmathi* (Debrecen)

1498:
Tenca, Luigi. La versiera di ... Guido Grandi. Boll. Un. Mat. Ital. (3) **12** (1957), 458–460. (English summary)

Riassunto dell'autore: "Si mostra ancora una volta, esaurientemente, quanto sia errato attribuire la curva versiera a Gaetana Agnesi che ne trattò in breve, nel 1748, nelle sue Instituzioni Analitiche, senza portare nessun nuovo contributo allo studio di detta curva, già inventata e studiata da Guido Grandi, fino dal 1703."

1499:
Murata, Tamotsu. French Empiricism. IV. Comment. Math. Univ. St. Paul. **8** (1960), 81–96.

Concluding remarks to the previous papers [same Comment. **6** (1958), 93–114; **7** (1959), 57–63; MR **19**, 1247; **21** #4078]. *A. Heyting* (Amsterdam)

1500:
Cartan, Henri. ★Nicolas Bourbaki und die heutige Mathematik. Arbeitsgemeinschaft für Forschung des Landes Nordrhein-Westfalen, Heft 76. Westdeutscher Verlag, Cologne and Opladen, 1959. 27 pp.

A spirited defense of the axiomatic method as followed by Bourbaki, together with the frankest exposition that the reviewer has seen from an authoritative source of Bourbaki's actual (as well as legendary) existence, methods of working, and objectives. Discussion by Behnke, Hoheisel, Peschl, Cremer and Cartan.
R. P. Boas, Jr. (Evanston, Ill.)

1501:
Joravsky, David. Soviet scientists and the great break. Daedalus **89** (1960), 562–580.

The article includes an account of the Moscow Mathematical Society during the years 1930–1932.

1502:
Kantorovič, L. V.; Natanson, I. P. Grigorii Mihaĭlovič Fihtengol'c. Necrologue. Uspehi Mat. Nauk **14** (1959), no. 5 (89), 123–128. (1 plate) (Russian)

1503:
Kuroš, A. G. Anatoliĭ Ivanovič Mal'cev (on his fiftieth birthday). Uspehi Mat. Nauk **14** (1959), no. 6 (90), 203–211. (1 plate) (Russian)

A general and scientific biography, with one photograph and a bibliography of 59 titles.

1504:
List of papers of the late J. Egerváry. Mat. Lapok **10** (1959), 1–4. (Hungarian)

1505:
Szénássy, Barna. The work of the late Z. Geőcze. Mat. Lapok **10** (1959), 26–38. (Hungarian)

1506:
Miranda, Carlo. Renato Caccioppoli. Ann. Mat. Pura Appl. (4) **47** (1959), v–vii.

A brief general and scientific biography.

1507:
Sauer, Robert; Heinrich, Helmut. Friedrich Adolf Willers: Sein Leben und Wirken. Z. Angew. Math. Mech. **40** (1960), 1–8. (1 plate)

1508:

Peirce, Charles Sanders. ★Collected papers. Edited by Charles Hartshorne and Paul Weiss. 6 vols. I: Principles of philosophy. II: Elements of logic. III: Exact logic. IV: The simplest mathematics. V: Pragmatism and pragmaticism. VI: Scientific metaphysics. The Belknap Press of Harvard University Press, Cambridge, Mass., 1960. xvi+393 pp., xii+535 pp., xiv+433 pp., x+601 pp., xii+455 pp., x+462 pp. Bound in double volumes. $45.00.

These volumes are a second printing of the same books that were published between the years of 1931 and 1935. The second printing is by offset from the first edition. Each volume of the second printing includes an errata on the occasional errors that appeared in the first edition. The full-page photographic plates appearing at the beginning of each volume in the first edition are omitted in the second printing.

The reviewer is perplexed by the fact that so many logicians and philosophers hold Peirce in such high esteem, but so few of them have ever read even a line of Peirce's works. The reviewer believes, probably over-optimistically, that the new availability of Peirce's works will correct this matter. *A. A. Mullin* (Urbana, Ill.)

LOGIC AND FOUNDATIONS

See also 1494, 1499, 1508, 1611.

1509:

Mihăilescu, Eugen. Formules normales des foncteurs de la logique classique. Acad. R. P. Romîne. Stud. Cerc. Mat. **10** (1959), 117–144. (Romanian. Russian and French summaries)

This is a continuation of the author's paper in same Stud. **2** (1951), 1–44 [MR **16**, 554]. The author adjoins to the axioms for E, A, R (there considered in § 2 of Chapter 2) axioms which express (in a certain rather peculiar way) the definitions of all 16 possible binary truth functors in terms of those three, and shows that the enlarged system has properties similar to those shown for the original one. I.e., the axioms are sufficient for all tautologies, but are not complete; there are infinitely many "free forms", all equivalent to one another; and the decision question is answered by a series of normal forms to one of which an arbitrary formula can be reduced.

H. B. Curry (University Park, Pa.)

1510:

Rasiowa, H.; Sikorski, R. On the Gentzen Theorem. Fund. Math. **48** (1959/60), 57–69.

According to the Gentzen Theorem for a formalization of the predicate calculus, which need not admit modus ponens as a primitive rule of inference, the deducibility of the formulas X and $X \rightarrow Y$ (from certain assumptions) entails the deducibility of the formula Y (from the same assumption). As (under the customary semantic rules) the validity of X and $X \rightarrow Y$ entails the validity of Y, the Gentzen Theorem must clearly hold true for every formalization which is (sound and) complete. The present authors take advantage of this connection by establishing, for a certain variant of Kanger's formalization of the predicate calculus, first the (soundness and) completeness theorem and then the Gentzen Theorem. It should, however, be observed that their model-theoretic proof is less satisfactory than extant finitary proofs as it does not produce a procedure for converting a given deduction of X and of $X \rightarrow Y$ into a deduction of Y.

E. W. Beth (Amsterdam)

1511:

Reichbach, Juliusz. Some definition of the proper prime ideal. Riveon Lematematika **13** (1959), 29–31.

This paper is an emendation of a previous paper which originally appeared in Polish with a rather extensive summary in English [Studia Logica **2** (1955), 229–250, 213–228; MR **17**, 446, 1172]. The author states without proof that the notion of "proper prime ideal" can be defined in terms of Gödel numberings, using certain sequences of formulas, and then draws the conclusion concerning the place of the proper prime ideal in the classification of Mostowski [Kwartalnik Filozoficzwy **16** (1946), 223–277; MR **9**, 129].

H. B. Curry (University Park, Pa.)

1512:

Słupecki, Jerzy. Towards a generalized mereology of Leśniewski. Studia Logica **8** (1958), 131–163. (Polish and Russian summaries)

Leśniewski's mereology is essentially poorer than other systems of set theory; in particular, it provides no basis for developing the arithmetic of natural numbers. In this paper Leśniewski's system is enriched so as to be suitable for laying the foundations of mathematics to the same extent as other systems of set theory. This is achieved by including certain new definitions and by using stronger logical methods; the basic intuitive assumptions of the original system are retained.

As a basis, the author uses the system S of the simple theory of types. Let S_1 be that part of S obtained by admitting only atomic expressions involving individual variables and variables for functors of one argument; it is known that S_1 makes allowance for the foundation of mathematics. On the other hand, the system of generalized mereology is obtained by adding to S the primitive term "O" and the axiom: $\sum_x (x=O)$, which expresses the fact that O is an arbitrarily chosen but determined object. Now a one-to-one relation is established between all expressions of S_1 and part of the expressions of generalized mereology. These last expressions form a system M which is also characterized as a separate system. Because M is isomorphic to S_1 it can be used to lay the foundations of mathematics. *E. W. Beth* (Amsterdam)

1513:

Fraenkel, Abraham A. ★Mengenlehre und Logik. Erfahrung und Denken: Schriften zur Förderung der Beziehungen zwischen Philosophie und Einzelwissenschaften, Bd. 2. Duncker & Humblot, Berlin, 1959. 110 pp. Paperbound: DM 18.60.

Die Sammlung "Erfahrung und Denken", in der diese Schrift erschienen ist, hat sich das hohe Ziel gesteckt, die Einzelwissenschaften und die Philosophie zu der Einheit der wissenschaftlichen Philosophie zusammenzuschmelzen und demgemäss zu einer philosophisch fundierten und auf den Einzelwissenschaften fussenden Weltanschauung zu

gelangen. Speziell hat die vorliegende Schrift zum Zweck, die Beziehungen zwischen der mathematischen Wissenschaft der Mengenlehre und der Logik herauszuarbeiten. Niemand ist dazu berufener als Professor Fraenkel, der eine Reihe höchst populärer Bücher über die Mengenlehre veröffentlicht hat. Die Schrift ist für mathematisch und logistisch nicht vorgebildete Leser bestimmt, und ist demgemäss ausführlich und möglichst einfach gehalten.

Inhaltsangabe: In der Einleitung wird die historische Entstehung des zentralen Begriffs der Mengenlehre, des aktual abgeschlossenen Unendlichen, geschildert. Im § 2 werden die Begriffe der Menge, Teilmenge, Summe und Durchschnitt, Abbildung, Äquivalenz und Kardinalzahl entwickelt. Dabei wird das Wesen des von Russell eingeführten logischen "Principle of abstraction" erläutert. Im § 3 werden die Begriffe der abzählbaren Menge und des Kontinuums eingeführt und mit logischen Bemerkungen, z.B. über Unmöglichkeitsbeweise, versehen. Im § 4 wird das Auswahlprinzip hinsichtlich seiner logischen Natur behandelt. Im fünften Paragraphen wird die Existenz unendlich vieler verschiedener transfiniter Kardinalzahlen bewiesen, eine Anzahl Antinomien der Mengenlehre besprochen und die Gelegenheit wahrgenommen, auf die damit zusammenhängenden logischen und philosophischen Probleme, z.B. im Zusammenhang mit dem sogenannten "quasi-kombinatorischen" Prozess von Bernays, einzugehen. Schliesslich, da man "zu einer hinreichenden Bewertung der Mengenlehre, auch vom logischen Standpunkt aus, nicht gelangen kann, ohne die wichtigsten ihrer eigenen Problemstellungen zu skizzieren", wurden noch zwei Paragraphen über die Operationen mit Kardinalzahlen (§ 6) und über die Ordnung und das Problem der Wohlordnung, hinzugefügt (§ 7). Dabei wird an den Leser allerdings in acht Sätzen über die Wohlordnung die weitgehende Forderung der strengen logischen Unterscheidung verwandter Begriffe gestellt. Zum Schluss des 7. Paragraphen wird der Wohlordnungssatz bewiesen und die Gleichwertigkeit des Auswahlprinzips und des Wohlordnungssatzes dargelegt. Der Schluss der Schrift bringt einiges über die Bedeutung der Mengenlehre für Mathematik und Logik und es wird dabei die stattgefundene "Tieferlegung der Fundamente", zwecks der mengentheoretischen Begründung der verschiedenen Zweige der Mathematik, erwähnt.

B. Germansky (Berlin)

1514:

Kripke, Saul A. A completeness theorem in modal logic. J. Symb. Logic **24** (1959), 1–14.

The author considers a first order predicate calculus with equality based on the Lewis modal propositional calculus S5. He gives a definition of a universally valid formula for the resulting calculus, which he names $S5^{*-}$, and develops complete formalisations of this calculus. Such formalisations use two primitive rules of procedure and the axiom schemes are obtained from any complete set for the classical predicate calculus by adjoining a further three.

The author also shows that, if the primitive symbols ( ) and = are omitted, a complete formalisation of S5 is obtained. He shows further that if a formula A of S5 with propositional qualifiers is unprovable in that system, then adding the closure of A renders the system inconsistent.

A. Rose (Nottingham)

1515:

Tugué, Tosiyuki. Predicates recursive in a type-2 object and Kleene hierarchies. Comment. Math. Univ. St. Paul. **8** (1960), 97–117.

After an introduction of notations and terminology in Section 1, the author introduces in Section 2 the type-2 predicate $\mathbf{E}_1 : \mathbf{E}_1(\alpha) = 0$ if $(\beta)(Ex)[\alpha(\bar{\beta}(x)) = 0]$, and $= 1$ otherwise. In Section 3 the author considers forms $(E\beta)(x)R(a, \bar{\beta}(x))$ and $(\beta)(Ex)R(a, \bar{\beta}(x))$ where $\bar{\beta}(x) = \prod_{i<x} p_i^{\beta(x)+1}$ and p_i is the $(i+1)$st prime. It is shown that a predicate expressible in one of the one-(type-1)-quantifier forms is not in general expressible in one of the above forms with $R(a, u)$ primitive (or general) recursive. Further, a general recursive predicate is not in general expressible in one of the above forms with $R(a, u)$ primitive recursive. Finally, there is no enumeration theorem for predicates of the second form above with $R(a, u)$ primitive (or general) recursive. It is shown in Section 4 that for each $y \in O_{2_0}$, the predicate $\mathfrak{H}_y(a)$ given by Addison and Kleene [Proc. Amer. Math. Soc. **8** (1957), 1002–1006; MR **19**, 934] is general recursive in $\mathbf{E}_1$. In Section 5 the author shows that predicates of variables of types 0, 1 expressible in one of the two-(type-1)-quantifier forms are exactly those expressible by prefixing a type-1 quantifier to a predicate general recursive in $\mathbf{E}_1$. That no second separation result for classes of sets definable by the one-(type-1)-quantifier predicates with free variables of type 2 exists is shown in Section 6. In Section 7, the author extends the notion of general recursive degrees and introduces a hyperanalytic hierarchy.

E. J. Cogan (Bronxville, N.Y.)

1516:

Simon, Herbert A. Definable terms and primitives in axiom systems. The axiomatic method. With special reference to geometry and physics. Proceedings of an International Symposium held at the Univ. of Calif., Berkeley, Dec. 26, 1957–Jan. 4, 1958 (edited by L. Henkin, P. Suppes and A. Tarski), pp. 443–453. Studies in Logic and the Foundations of Mathematics. North-Holland Publishing Co., Amsterdam, 1959. xi+488 pp. $12.00.

Die Abhandlung beschäftigt sich mit Definierbarkeitsfragen für axiomatisierte Theorien der empirischen Wissenschaften. Der Verf. schlägt gewisse Abschwächungen der inzwischen schon fast klassisch gewordenen Definition der Definierbarkeit, die von A. Tarski stammt, vor, um damit Anforderungen zu erfüllen, die von empiristischer Seite an die Axiomatisierung naturwissenschaftlicher Theorien gestellt worden sind. Fragen dieser Art sind schon seit langem diskutiert worden (z.B. von R. Carnap), ohne dass bisher eine vollauf befriedigende Lösung gefunden wäre.

H. Kiesow (Oberhausen)

SET THEORY

1517:

Läuchli, Hans. Mischsummen von Ordnungszahlen. Arch. Math. **10** (1959), 356–359.

If Φ is a sequence of ordinal numbers, let Φ^* denote the subsequence of its nonzero terms. With the ordinal number α whose normal form is $\omega^{\alpha_1} + \cdots + \omega^{\alpha_1}$ and the

natural number $r \geq k$, associate the set $F(\alpha, r)$ of those r-termed sequences Φ of ordinal numbers for which $\Phi^* = (\omega^{\alpha_k}, \cdots, \omega^{\alpha_1})$. For ordinal numbers $\xi^{(1)}, \cdots, \xi^{(n)}$, $\bigoplus_{i=1}^{n} \xi^{(i)}$ is defined to be $\sum \xi^{(i)}$ or $\max \xi^{(i)}$ according as all $\xi^{(i)} \leq 1$ or not. Given ordinal numbers α_i $(i = 1, \cdots, n)$ with corresponding normal forms $\omega^{\alpha_{i1}^{(i)}} + \cdots + \omega^{\alpha_{ik_i}^{(i)}}$, set $r = \sum_{i=1}^{n} k_i$. Neumer [Arch. Math. **5** (1954), 244–248; MR **16**, 19] established the existence and determined the form of the largest and smallest "mixed sums" of $\alpha_1, \cdots, \alpha_n$. The author shows that μ is a mixed sum of $\alpha_1, \cdots, \alpha_n$ if, and only if, there exist sequences $(\xi_1^{(i)}, \cdots, \xi_r^{(i)}) \in F(\alpha_i, r)$ $(i = 1, \cdots, n)$ with $\mu = \sum_{j=1}^{r} (\bigoplus_{i=1}^{n} \xi_j^{(i)})$. It follows that the ordinals $\alpha_1, \cdots, \alpha_n$ have only a finite number of mixed sums. *F. Bagemihl* (South Bend, Ind.)

1518:

Ricabarra, Rodolfo. Partitions in sets of ordinal numbers. Rev. Mat. Cuyana **2** (1956), 1–27 (1958). (Spanish)

The paper deals with the space $I = I\omega_1$ of ordinals $< \omega_1$ and in particular with regressive functions ($fx < x$) whose domain and range $\subseteq I$ and with partitions of I. Let any set $A \subseteq I$ be called a p-set provided it intersects each closed set in I of cardinality $\aleph_1$; let n-set mean non-p-set [cf. also W. Neumer, Math. Z. **54** (1951), 254–261; MR **13**, 331; G. Bloch, C. R. Acad. Sci. Paris **236** (1953), 265–268; MR **14**, 733; G. Fodor, Acta Sci. Math. Szeged **17** (1956), 139–142; MR **18**, 551]. If F is a family of disjoint sets $\subseteq I$, then $\bigcup F$ is an n-set if and only if each $Y \in F$ and the set X of the first points in the Y's are n-sets (Th. 2). If the domain $D \subset I$ of a regressive function f is a p-set, then at least for one $\alpha \in fD$ the set $\{f_\alpha^{-1}\}$ is a p-set (Th. 3). The main theorem (7) deals with partitions of I and constructions of ramified sequences (tables where each element has comparable points in each row) in the sense of the reviewer [Thesis, Paris, 1935 or see Acta Math. **75** (1943), 139–150; MR **7**, 195]. If F is a partition of I, then by iteration of F one gets a system of subpartitions yielding a ramified table or tree C_F. On the other hand, for each pair of non-countable subsets G, A of I, one defines $A^0 = A$, $A^{\alpha+1} = T_{A^\alpha}(G)$, where T_{A^α} means the isomorphism from I to A^α provided A^α be non-countable; $A^\lambda = \bigcap_{\beta<\lambda} A^\beta$; the order type of the sequence of non-void sets G^α (defined writing G instead of A) is denoted by $on(G)$. The ramified sequences $C(\alpha, j)$ consist of countable sequences, each sequence x being the juxtaposition of finitely many constant sequences with value $\beta < \alpha$; the length of this interval is $< j(\beta)$, $j(\beta)$ being an ordinal of the form ω^β (and ω_1 in particular). By ordering each knot of the table one associates to C_F a set $\bar{\varphi}C_F$ of sequences of ordinals $< \omega_1$ ordered by prolongation of sequences. The main theorem (7) consists in the equality $\bar{\varphi}C_F = C(\alpha, j)$, where $\alpha = o(F)$ denotes the order type of the set of first elements of elements of partition $I \to F^\beta I$; $F^\beta I$ is the βth term of the partition; $j(\beta) = on(F^\beta)$. Each $C(\alpha, j)$ is generated in this way. The proof of Theorem 7 takes 11 pages (16–26).

Đ. Kurepa (Princeton, N.J.)

1519:

Ore, Oystein; Motzkin, T. S. Subsets and subgraphs with maximal properties. Proc. Amer. Math. Soc. **10** (1959), 965–969.

Let G be a given set, $\{F_i\}$ be finite subsets of G, and H a maximal subset of G not containing any F_i. The author (O. Ore) defines a property of such sets H called "strong simplicity" and shows that if all H enjoy this property, they are equinumerous. As an application of these set-theoretic notions, he proves that if G is a connected graph regarded as a set of lines and each F_i is bridgeless then any maximal subgraph H is connected and covers the vertices of G. Also, if T_1 and T_2 are any two maximal subtrees of an infinite graph G, then there is a finite sequence of maximal subtrees beginning with T_1 and ending with T_2, each term of which after the first is obtained from the preceding one by deleting one line and adding another. (This result is well known in the finite case.) In a concluding section, T. S. Motzkin considers the situation when the forbidden subsets F_i are infinite.

F. Harary (Ann Arbor, Mich.) and *J. Myhill* (Stanford, Calif.)

COMBINATORIAL ANALYSIS

See also 1519, 1628, B2494.

1520a:

Pták, Vlastimil. On a combinatorial theorem and its application to nonnegative matrices. Czechoslovak Math. J. **8** (**83**) (1958), 487–495. (Russian. English summary)

1520b:

Pták, Vlastimil; Sedlaček, Irži [Sedláček, Jiří]. On the index of imprimitivity of nonnegative matrices. Czechoslovak Math. J. **8** (**83**) (1958), 496–501. (Russian. English summary)

Let $N = \{1, 2, \cdots, n\}$. Let F be the set of mappings ϕ such that $A_1 \subset N$, $A_2 \subset N \Rightarrow \phi(A_1) \subset N$ and $\phi(A_1 \cup A_2) = \phi(A_1) \cup \phi(A_2)$, with ϕ of the empty set being the empty set. Let $\phi = 0$ mean $\phi(A)$ empty for each $A \subset N$. Call ϕ reducible if for some $P \subset N$ such that $0 \neq P \neq N$, $\phi(P) \subset P$. Otherwise, ϕ is irreducible. Let $\phi \in F$ be irreducible; then $\phi(N) = N$ and $\phi(A)$ is nonempty when A is nonempty. If A is nonempty, then $A \cup \phi(A) \cup \cdots \cup \phi^{n-1}(A) = N$. Either $\phi, \phi^2, \cdots, \phi^n$ are irreducible, in which case ϕ^v is irreducible for each v and $\phi^p(A) = N$ for each nonempty A and $p \geq 1 + (n-1)^2$; or ϕ^k is reducible for some $k \leq n$. In the latter case, ϕ^v is reducible for infinitely many v, and for such v there is a $d > 1$, $d \mid v$, and a nonempty set P such that the sets $P_j = \phi^j(P)$, $0 \leq j \leq d-1$, are mutually disjoint and (1) $P_0 \cup P_1 \cup \cdots \cup P_{d-1} = N$, (2) $\phi^d(P_i) = P_i$, $0 \leq i \leq d-1$, and (3) $Q \subset N$, $\phi^r(Q) \subset Q \Rightarrow Q$ is the union of some P_i. The sets $A_0, A_1, \cdots, A_{s-1}$ constitute a cyclic partition of N if $A_j = \phi(A_{j-1})$, $1 \leq j \leq s-1$ and $A_0 = \phi(A_{s-1})$ and $A_0, \cdots, A_{s-1}$ is a partition of N. s is the length of the partition. Let $A_0, \cdots, A_{s-1}$ be a cyclic partition of N. If $x \in \phi^j(x)$, then $s \mid j$. From this it follows that among the cyclic partitions of N relative to ϕ there is one of maximum length. This length $r(\phi)$ is the index of imprimitivity of ϕ. If $r(\phi) = 1$, then ϕ is said to be primitive. If $\Phi = \phi^v$, then Φ is irreducible on each A_j. $\Phi(Q) \subset Q \Rightarrow Q$ is the union of some A_j and $\Phi(Q) = Q$. If $s = r(\phi)$, then Φ is primitive on each A_j. If $s = r(\phi)$ and $k \geq 1 + (n-s)^2$, then $x \in \phi^{ks}(x)$ for each x. The sequence $x_0, x_1, \cdots, x_q$ of elements of N is said to be a cycle of ϕ if $x_i \in \phi(x_{i-1})$,

$1 \leq i \leq q$ and $x_0 = x_q$ and $x_i \neq x_j$ for $0 \leq i < j < q$. The length of the cycle is q. The main result is that the greatest common divisor of all lengths of cycles of ϕ is $r(\phi)$.

S. Sherman (Philadelphia, Pa.)

1521:

Munch, Ove J. On power product sums. Nordisk Mat. Tidskr. **7** (1959), 5–19. (Danish)

The expansion of arbitrary polynomials of degree n relative to the basis

$$B = \left\{ \binom{x}{n}, \binom{x+1}{n}, \cdots, \binom{x+n}{n} \right\}$$

is studied. If $P(x)$ and $P^*(x)$ satisfy

$$P(x-1) = (-1)^n P^*(-x)$$

and if

$$P(x) = \sum_{i=0}^{n} a_i \binom{x+i}{n}$$

then

$$P^*(x) = \sum_{i=0}^{n} a_{n-i} \binom{x+i}{n}.$$

In the definition $\Lambda(n; q) = \sum_K n \, k_1^{p_1} k_2^{p_2} \cdots k_q^{p_q}$, summation is over all sets k_i with $1 \leq k_1 (\leq) k_2 (\leq) \cdots (\leq) k_q \leq n$, where K refers to a specific choice among the 2^{q-1} ways of specifying which inequalities are strict. Let $\sum_{i=1}^{q} p_i = p$. It is proved that $\Lambda(n; q)$ is a polynomial in n of degree $p+q$, with leading coefficient $\prod_{\nu=1}^{q} (\nu + \sum_{i=1}^{\nu} p_i)^{-1}$, and integer zeros at $m, m-1, \cdots, m-q$. If K^* differs from K by interchange of '$<$' and '$\leq$', and $\Lambda^*(n; q)$ is summed over K^*, then

$$\Lambda(n-1; q) = (-1)^{p+q} \Lambda^*(-n; q).$$

In the remainder of the article, identities relating the Euler numbers, Stirling numbers, and Bernoulli numbers to the basis B are derived.

S. W. Golomb (Pasadena, Calif.)

1522:

Mitrinović, Dragoslav S.; Djoković, Dragomir. Sur une relation de récurrence concernant les nombres de Stirling. C. R. Acad. Sci. Paris **250** (1960), 2110–2111.

Put $x(x-1)\cdots(x-n+1) = \sum_{r=0}^{n} S_n^r x^r$, $S_0^0 = 1$, $S_n^0 = 0$ $(n \geq 1)$, $S_n^r = 0$ $(r > n)$. The authors prove the formula

$$(-1)^{n-m} S_n^m = \frac{1}{n-m} \sum_{r=0}^{n-m-1} (-1)^r \binom{n-r}{m-1} S_n^{n-r} \quad (m < n).$$

L. Carlitz (Durham, N.C.)

1523:

Harary, Frank. On the group of the composition of two graphs. Duke Math. J. **26** (1959), 29–34.

This paper deals with the structure of the automorphism groups of various products of graphs. Given a graph G, denote its sets of vertices and edges by P and L, respectively. All graphs considered are finite. A lexicographic product of two graphs G_1, G_2, called the composition and denoted by $G_1[G_2]$, is defined by $P = P_1 \times P_2$, $L = \{(a_1, b_1)(a_2, b_2) : a_1 a_2 \in L_1$, or $a_1 = a_2$ and $b_1 b_2 \in L_2\}$, and the automorphism group Γ of this graph is investigated. It is stated on p. 32 that if G_1 and G_2 are not both complete, then $\Gamma(G_1[G_2]) = \Gamma(G_1)[\Gamma(G_2)]$, the "Gruppenkranz" of $\Gamma(G_1)$ and $\Gamma(G_2)$ [see Pólya, Acta Math. **68** (1937), 145–254]. This is, however, not correct, a counterexample being $G_1 = K_n$, $n \geq 2$, G_2 any graph with G_2' disconnected. Also counterexamples can be given such that neither G_1 nor G_2 is complete. A minor error on p. 32, lines 7 and 12, is that K_{m+n} should be K_{mn}; correspondingly, in lines 12 and 13, $\mathfrak{S}_{m+n}$ should be $\mathfrak{S}_{mn}$. The groups of three other products of graphs (the direct product, the join, and the cartesian product) are also briefly discussed. Throughout, the emphasis is on permutation groups rather than on abstract groups.

G. Sabidussi (New Orleans, La.)

1524:

Sabidussi, Gert. The composition of graphs. Duke Math. J. **26** (1959), 693–696.

The author corrects an error of the reviewer [#1523] by proving a theorem whose specialization to finite graphs is as follows. Let G_1 and G_2 be two graphs whose groups are Γ_1 and Γ_2. Let $G = G_1 \circ G_2$ be the composition of the graphs G_1 and G_2 and let $\Gamma_1 \circ \Gamma_2$ be the composition of their groups. Let V_1 be the sets of points of G_1. The neighborhood $N(v)$ of a point $v \in V_1$ is the set of points adjacent to v in G_1. Let R and S be the equivalence relations on V_1 defined by uRv if $N(u) = N(v)$ and uSv if $N(u) \cup \{u\} = N(v) \cup \{v\}$. Then the group of the composition G of two graphs G_1 and G_2 is the composition $\Gamma_1 \circ \Gamma_2$ of their groups if and only if G_2 is connected if R is not the identity relation and the complement of G_2 is connected if S is not the identity relation.

F. Harary (Ann Arbor, Mich.)

1525:

Ore, Oystein. Graphs and subgraphs. II. Trans. Amer. Math. Soc. **93** (1959), 185–204.

This is a continuation of an earlier paper by the author [same Trans. **84** (1957), 109–136; MR **18**, 751]. Given a graph G, the author considers the system of alternating paths starting from a specified vertex c_0 and defined in terms of a specified subgraph H. He shows that many properties of this system are "invariant"; that is, they depend on the local degrees of H but not otherwise on its structure. In particular the bicursal blocks, the components of the subgraph of bicursal edges, are invariant. In chapter 6 the author obtains a condition for a regular graph G_n of degree n to be decomposable into n factors of the first degree. It is required that G_n shall have a known factor H of the first degree such that each edge of G_n belongs to an alternating cyclic path with respect to H. The condition is then stated in terms of a corresponding system of bicursal blocks.

W. T. Tutte (Toronto)

1526:

Sedláček, Jiří. Eine Bemerkung über Euler'sche Graphen. Mat.-Fyz. Časopis. Sloven. Akad. Vied **8** (1958), 151–154. (Czech. Russian and German summaries)

In dieser Bemerkung zeigt der Verf., wie man die von O. Ore [Elem. Math. **6** (1951), 49–53; MR **12**, 845] und F. Bäbler [Comment. Math. Helv. **27** (1953), 81–100; MR **15**, 50] behandelte Klasse von Eulerschen Graphen G mit einem auf sämtlichen Kreisen von G liegenden Knotenpunkt verallgemeinern kann. (Wegen analogischer Fragen für gerichtete Graphen s. eine frühere Arbeit des Verf. [Časopis Pěst. Mat. **84** (1959), 7–15; MR **21** #5202].)

M. Fiedler (Prague)

ORDER, LATTICES

See also 1837.

1527:

Szász, Gábor. ★Bevezetés a hálóelméletbe. [Introduction to lattice theory.] Akadémiai Kiadó, Budapest, 1959. 225 pp. 60 Ft.

This is an expository text on lattice theory and as such it has the same aim as H. Hermes's *Einführung in die Verbandstheorie* [Springer-Verlag, Berlin, 1955; MR **16**, 667]. On the whole it covers the ground more fully; an exception is a part of the discussion on projective geometries. The very clear, simple and elegant presentation makes the book easy reading even for the complete beginner. At the same time a considerable amount of additional information, with references instead of proofs, is supplied in small print for the more advanced student of the subject. A wide range of examples demonstrates the connections of lattice theory with other branches of mathematics. Each chapter concludes with a number of exercises (some with hints for solution) to help assimilate the material. A large bibliography and a subject index are also given.

The following (incomplete) list may serve to indicate the subject matter treated. I. Partially ordered sets; including also chain conditions, Zorn's lemma, dimension functions. II. On lattices in general: fundamental concepts and a discussion of the independence of lattice axioms. III. Complete lattices: conditionally complete lattices, substructure-lattices of general algebraic structures, closure operations, Galois connections, Dedekind cuts, partially ordered sets as topological spaces. IV. Distributive and modular lattices; including also infinitely distributive and completely distributive lattices. V. Certain classes of modular lattices: modular lattices of locally finite length, valuations of and metrics on lattices, complemented modular lattices and projective geometries. VI. Boolean algebras: standard results, the algebra of relations on a set, the algebra of statements, valuations of Boolean algebras. VII. Semi-modular lattices; including also a discussion of general linear dependence. VIII. Ideals in lattices; including also the isomorphic representation of distributive lattices as rings of sets, and of Boolean algebras as fields of sets. IX. Congruence relations, direct and subdirect decompositions of algebraic structures in general and of lattices in particular; general form of the Schreier refinement theorem; connection between ideals and congruence relations of a lattice; neutral elements; the centre.

It is welcome news that a German edition of the book is being prepared. *L. G. Kovács* (Keele)

1528:

Naito, T. On a problem of Wolk in interval topologies. Proc. Amer. Math. Soc. **11** (1960), 156–158.

The author considers partially ordered sets in which every set of pairwise incomparable elements is finite. He shows that if a net f and all its subnets have a common least upper bound y for their ranges, then there is a directed subset of the range of f with least upper bound y. It follows that the order and interval topologies coincide. As the author notes, the condition on sets of incomparable elements is necessary for coincidence of the topologies in the case of complete lattices, so that his result is nearly best possible. *J. Isbell* (Seattle, Wash.)

1529:

Słowikowski, W. Metric semilattices and their connections with the measure theory. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **7** (1959), 111–117. (Russian summary, unbound insert)

Ein Halbverband ist eine teilweise geordnete Menge, in welcher jede endliche Teilmenge ein Supremum hat. Ein Halbverband L heisst pseudometrisch, wenn auf L eine Pseudometrik ρ erklärt ist, für welche $\rho(a \vee c, b \vee c) \leqq \rho(a, b)$ bei jedem $a, b, c \in L$ richtig ist. Wenn

$$\rho(a \vee b \vee c, c) = \rho(a \vee b \vee c, b \vee c) + \rho(b \vee c, c)$$

für jedes $a, b, c \in K$ gilt, wobei K eine Teilmenge von L ist, so wird die Pseudometrik ρ additiv auf K genannt. Ein pseudometrischer Halbverband L heisst eigentlich, wenn jede ordnungskonvergente monotone Fundamentalfolge in L auch pseudometrisch konvergent zum selben Limes ist.

Es werden u. a. folgende Ergebnisse mitgeteilt: Ist K ein Teilhalbverband eines pseudometrischen Halbverbandes L und ist die Pseudometrik ρ auf K additiv, so hat sie auch auf der abgeschlossenen Hülle von K die selbe Eigenschaft. Ist L eigentlich und ρ auf einer monotonen ordnungskonvergenten Folge additiv, so ist die Folge auch pseudometrisch konvergent gegen denselben Limes. Ist L_0 eine abgeschlossene Teilmenge eines eigentlichen pseudometrischen Halbverbandes, auf der ρ additiv ist, so enthält L_0 auch den Ordnungslimes jeder monotonen ordnungskonvergenten Folge von L_0.

Ist auf einem Mengenkörper $\mathfrak{M}$ ein Mass gegeben, so bilden alle zu $\mathfrak{M}$ gehörenden Mengen von endlichem Mass einen pseudometrischen Halbverband, wenn der Abstand zweier Mengen als Mass ihrer symmetrischen Differenz definiert ist. Durch Anwendung der oben angeführten Ergebnisse kann ein auf $\mathfrak{M}$ erklärtes Mass zu einem auf $\overline{\mathfrak{M}}$ erklärten Mass erweitert werden, wo $\overline{\mathfrak{M}}$ der kleinste σ-Körper über $\mathfrak{M}$ ist. Ähnlich wird auch die Erweiterung eines Integrals behandelt. *M. Novotný* (Brno)

1530:

Amemiya, Ichiro; Halperin, Israel. Complemented modular lattices. Canad. J. Math. **11** (1959), 481–520.

I. Kaplansky hat bewiesen [Ann. of Math. (2) **61** (1955), 524–541; MR **19**, 524], dass ein vollständiger orthokomplementärer modularer Verband stetig ist. Die Verfasser untersuchen meistens die Bedingungen, aus welchen die $\aleph$-Stetigkeit eines $\aleph$-vollständigen modularen Verbandes folgt; dabei ist $\aleph$ eine beliebige unendliche Mächtigkeit und die $\aleph$-Vollständigkeit und $\aleph$-Stetigkeit stellt eine natürliche Verallgemeinerung der Vollständigkeit bzw. Stetigkeit dar, die darin besteht, dass nur die Teilmengen des Verbandes, deren Mächtigkeit $\leqq \aleph$ ist, den entsprechenden Definitionsforderungen genügen.

Zu diesen Zwecken werden drei verschiedene Modifikationen des Begriffes einer unabhängigen Teilmenge in einem $\aleph$-vollständigen modularen Verbande und die Endlichkeit des Verbandes eingeführt: Ein $\aleph$-vollständiger komplementärer modularer Verband heisst endlich,

wenn in jeder unabhängigen Folge zueinander perspektiver Elemente dieses Verbandes alle Elemente gleich dem Nullelement 0 sind. Die Endlichkeit eines $\aleph$-vollständigen komplementären modularen Verbandes und die Existenz eines gewissen Antiautomorphismus (der eine Verallgemeinerung der Orthokomplementarität darstellt) zieht die $\aleph$-Stetigkeit des Verbandes nach sich. Da aus der Orthokomplementarität die Endlichkeit folgt, ist jeder $\aleph$-vollständige orthokomplementäre modulare Verband $\aleph$-stetig.

Diesen Hauptergebnissen geht eine systematische Untersuchung aller eingeführten Unabhängigkeitsbegriffe voran; z.B. wird ein Satz über die Bildung von sogenannten streng unabhängigen Systemen aus gegebenen streng unabhängigen Systemen in einem $\aleph$-vollständigen modularen Verband mit Nullelement bewiesen. Sind in einem Verband dieser Art zwei streng unabhängige Systeme $\{a_\lambda\}$, $\{b_\lambda\}$ so gegeben, dass a_λ zu b_λ perspektiv ist, dann gibt es unter gewissen Voraussetzungen eine perspektive Abbildung des Intervalls $[0, \bigvee a_\lambda]$ auf $[0, \bigvee b_\lambda]$, bei welcher b_λ ein Bild von a_λ ist. Sind in einem $\aleph$-vollständigen komplementären modularen Verband die Intervalle $[0, a]$, $[0, b]$ endlich bzw. stetig, so hat auch das Intervall $[0, a \vee b]$ dieselbe Eigenschaft. Ausführlich werden auch die sogenannten homogenen Folgen eines $\aleph$-vollständigen komplementären modularen Verbandes studiert: das sind streng unabhängige Folgen, deren Elemente zueinander perspektiv sind. Es wird z.B. bewiesen: Ist $\{a_n\}$ eine homogene Folge, so ist der Verband $[0, \bigvee a_n]$ genau dann nach oben $\aleph_0$-stetig, wenn alle a_n ein gemeinsames Komplement haben. *M. Novotný* (Brno)

1531:

Murata, Kentaro. Decomposition of radical elements of a commutative residuated lattice. J. Inst. Polytech. Osaka City Univ. Ser. A **10** (1959), 31–34.

[For the terminology see G. Birkhoff, *Lattice theory*, Amer. Math. Soc., New York, 1948; MR **10**, 673; chapter 13.] Let L be a commutative residuated cm-lattice with a greatest element e and suppose that $ab \leqq a$ for any $a, b \in L$. Define inductively $a^{(1)} = a$, $a^{(\rho)} = a^{(\rho-1)} \cdot a^{(\rho-1)}$ for $\rho > 1$. Let Γ_a be the set of elements x with $a \leqq x$ and $x^{(\rho)} \leqq a$ for a suitable integer $\rho = \rho(x)$. If $\sup[\Gamma_a] = a$, then a is called a radical element of L.

The following is the main result of the paper. If a is a radical element of L and if the ascending chain condition holds for the interval $[e, a]$, then a has a unique minimal decomposition as an intersection of prime elements.

This is a lattice-formulation of a result of E. Schenkman concerning groups [Proc. Amer. Math. Soc. **9** (1958), 375–381; MR **20** #60]. *Št. Schwarz* (Bratislava)

1532:

Avann, S. P. Upper and lower complementation in a modular lattice. Proc. Amer. Math. Soc. **11** (1960), 17–22.

The author defines the concept of upper and lower complements for elements of a finite modular lattice and proves that uniqueness of upper (or lower) complements implies distributivity. He also gives a number of equivalent conditions for uniqueness of upper (or lower) complements. *R. P. Dilworth* (Pasadena, Calif.)

1533:

Mihalek, R. J. Modularity relations in lattices. Proc. Amer. Math. Soc. **11** (1960), 9–16.

A definition is given of 'modularity relation', which contains as a special case modularity R relative to a join-closed subset S of a lattice L: bRc if and only if $b, c \in S$ and $(x \cup b) \cap c = x \cup (b \cap c)$ for all $x \leqq c$ in S. Various properties of abstract modularity relations, and neighboring concepts, are deduced from those postulated. *G. Birkhoff* (Cambridge, Mass.)

GENERAL MATHEMATICAL SYSTEMS

1534:

Ádám, András. On permutations of set products. Publ. Math. Debrecen **5** (1957), 147–149.

Let S be an abstract algebra in the sense of G. Birkhoff [Proc. Cambridge Philos. Soc. **31** (1935), 433–454]. Suppose that S has a unique one-element subalgebra and is the Cartesian product of subalgebras S_t ($t \in T$). The automorphism group G of S has subgroups $G_1 \supseteq G_2 \supseteq G_3$ defined as follows: $G_1 = \{\alpha \in G |\ \alpha$ maps each S_t onto some $S_{t'}\}$; $G_2 = \{\alpha \in G |\ S_t\alpha = S_t$ for all $t \in T\}$; $G_3 = \{\alpha \in G |\ x\alpha = x$ for all $x \in \bigcup_{t \in T} S_t\}$. Then G_1 is a split extension of G_2 by a Cartesian product of symmetric groups, and G_2 is a split extension of G_3 by the Cartesian product of the automorphism groups of the S_t. These facts are established more generally for a group G of permutations on a Cartesian product of sets with a common distinguished element. The conditions imposed on G reflect obvious properties of the above automorphism group. *P. J. Higgins* (London)

1535:

Fujiwara, Tsuyoshi. On mappings between algebraic systems. Osaka Math. J. **11** (1959), 153–172.

The algebraic systems $\mathscr{A}$ under consideration have a set V of finitary compositions and a set A_V of composition-identities. If A_V is empty, one speaks of a φ_V-system. If V and W are two sets of finitary compositions, a set of formal identities

(1) $\xi_\mu(v(x_1, \cdots, x_N)) =$

$$\mathscr{P}_{t_\mu, v}(\xi_1(x_1), \cdots, \xi_1(x_N); \cdots; \xi_m(x_1), \cdots, \xi_m(x_N)),$$

where v is N-ary and where $\mathscr{P}_{t_\mu, v}(y_{11}, \cdots, y_{1N}; \cdots; y_{m1}, \cdots, y_{mN})$ is in the free φ_W-system generated by $y_{11}, \cdots, y_{mN}$, is called a set of basic mapping-formulas, and denoted by $P_{V,W}\{\xi_1, \cdots, \xi_m\}$. If $\mathscr{B}$ is a φ_W-system, the Cartesian product $\mathscr{B}^m$ can be made into a φ_V-system in such a way that a set of mappings $\psi_1, \cdots, \psi_m$ from a φ_V-system $\mathscr{A}$ into $\mathscr{B}$ satisfies the family $P_{V,W}\{\xi_1, \cdots, \xi_m\}$ if and only if the mapping $a \to (\varphi_1(a), \cdots, \varphi_m(a))$ is a homomorphism of $\mathscr{A}$ into $\mathscr{B}^m$. This is the "algebraic Taylor's expansion theorem". Its consequences and related results constitute the major portion of the paper. Example: If $\mathscr{A}$ is a φ_V-system, then any family of mappings $\psi_1, \cdots, \psi_m$ of $\mathscr{A}$ satisfying P is determined by its values at generators of $\mathscr{A}$; if $\mathscr{A}$ is free with generators $\{a_\rho\}$, there is a family of P-mappings taking on prescribed values at the a_ρ. A family P as above is called (A_V, B_W)-universal if for every

free A_V-system $\mathscr{A}$, every set of free generators for $\mathscr{A}$, and every set of prescribed values for P-mappings into a B_W-system $\mathscr{B}$, there is a family of P-mappings of $\mathscr{A}$ into $\mathscr{B}$ taking on these values at the free generators. If $P = P_{V,V}$ is (A_V, A_V)-universal for all A_V (V fixed), P is called absolutely universal. If P consists solely of formulas

$$(2) \qquad \xi_\mu(v(x_1, \cdots, x_N)) = v(\xi_\mu(x_1), \cdots, \xi_\mu(x_N)),$$

then P is absolutely universal. Conversely, if P is absolutely universal and at least one composition of V is not unary, then P has the form (2).

G. B. Seligman (New Haven, Conn.)

THEORY OF NUMBERS

See also 1556, B1937, B1938, B2028.

1536:

Inkeri, K.; Sirkesalo, J. Factorization of certain numbers of the form $h \cdot 2^n + k$. Ann. Univ. Turku. Ser. A I **38** (1959), 15 pp.

The authors give the complete factorization of $N = h \cdot 2^n + k$ for $n = 1(1)32$ and $(h, k) = (3, \pm 5)$, $(3, \pm 7)$ and $(5, \pm 3)$. They use a variety of methods of factoring and tests for primality. A few new variations of the converse of Fermat's theorem and the tests of Lucas are given: If p is any prime factor of N not dividing N/p (including the case $p = N$), then the equation $x^p + y^p + z^p = 0$ has no solution (x, y, z) in integers not divisible by p. Of the 192 numbers N factored, all but 12 are square-free. {However, the first case of Fermat's last theorem is proved [D. H. Lehmer and E. Lehmer, Bull. Amer. Math. Soc. **47** (1941), 139–142; MR **2**, 250] for all odd primes $p < 253747889$. Only 12 new cases of primes beyond this limit are given.}

D. H. Lehmer (Berkeley, Calif.)

1537:

Jeśmanowicz, L. Several remarks on Pythagorean numbers. Wiadom. Mat. (2) **1** (1955/56), 196–202. (Polish)

W. Sierpiński [Wiadom. Mat. (2) **1** (1955/56), 194–195; MR **21** #4126] showed that the equation $3^x + 4^y = 5^z$ has $x = y = z = 2$ as its only solution in natural x, y, z. The integers 3, 4, 5 constitute a triad of Pythagorean numbers, integral solutions of the equation $a^2 + b^2 = c^2$. The author proves that Sierpiński's result holds also for the following sets of Pythagorean numbers: $2n+1$, $2n(n+1)$, $2n(n+1)+1$ with $n = 2, 3, 4, 5$ ($n = 1$ is Sierpiński's case).

H. Halberstam (London)

1538:

Schinzel, A. On prime numbers such that the sums of the divisors of their cubes are perfect squares. Wiadom. Mat. (2) **1** (1955/56), 203–204. (Polish)

Fermat posed the following question to Wallis: Does there exist a natural number n such that $\sigma(n)$, the sum of the divisors of n, is a perfect square? The author proves in a simple manner that the only prime p for which $\sigma(p^3)$ is a perfect square, is 7. A related result due to V. Thébault [Mathesis **62** (1953), 120–129; MR **14**, 949] states that the only prime p for which $\sigma(p^4)$ is a perfect square, is 3.

H. Halberstam (London)

1539:

Erdős, P. Some remarks on Euler's φ function. Acta Arith. **4** (1958), 10–19.

Let $A(n)$ denote the number of solutions of $\phi(m) = n$. According to a long unsettled conjecture of R. D. Carmichael, there is no integer n for which $A(n) = 1$. In this paper it is shown that, if there exists an integer n with $A(n) = k$, there are infinitely many such integers. This is accomplished by proving that for infinitely many primes p, if $A(n) = k$, then also $A[n(p-1)] = k$. Another result is the following:

$$\liminf_{n\to\infty} \left[\max_{1 \le i \le f(n)} \phi(n+i) / \min_{1 \le j \le f(n)} \phi(n+j) \right] = 1$$

for any sequence $f(n)$ tending to infinity with n in such a way that $f(n) \le F(n)$, where $F(n)$ is a complicated function, which is shown to be the best possible dominant. The author corrects an earlier statement [Mat. Lapok **7** (1956), 10–17; MR **20** #4534; p. 15] of the following theorem. Let $\lim g(n)/\log\log n = 0$. Then there exists an infinite sequence $\{n_k\}$ such that for all i, $1 \le i \le g(n_k)$, we have $1 - \varepsilon_k \le \phi(n_k + i)/\phi(n_k + i - 1) < 1 + \varepsilon_k$, where $\varepsilon_k \to 0$ as $k \to \infty$. The above condition on $g(n)$ is stated to be the best possible also. Among other results presented without proof the following is noteworthy. Let $a_1, a_2, \cdots, a_k$ be any non-negative integers or infinity. Then there exists an infinite sequence of integers, $n_1 < n_2 < \cdots$ such that $\lim_{j\to\infty} h(n_j + i)/h(n_j + i - 1) = a_i$ for $i = 1, 2, \cdots, k$, where $h(n) = n^c f(n)$ is multiplicative and $f(n)$ satisfies the conditions: $f(p_m^e) \to 1$ as $p_m^e \to \infty$, and $\sum_m |f(p_m^{b_m}) - 1| = \infty$ for a certain sequence $b_m \ge 1$ as p_m runs through the sequence of primes. This theorem was previously proved by A. Schinzel [Bull. Acad. Polon. Sci. Cl. III **3** (1955), 415–419; MR **17**, 461] in the cases $h = \phi$ and $h = \sigma$, where $\sigma(n)$ is the sum of the divisors of n.

R. J. Levit (San Francisco, Calif.)

1540:

Steuerwald, Rudolf. Über die Gleichung $x^2 - Dy^2 = Cz$. Math. Z. **73** (1960), 382–385.

The equation (*) of the title has a non-trivial solution in rational integers x, y, z if and only if $C \neq 0$, and the congruence $u^2 \equiv D \pmod{C}$ is solvable (D not a square). The author discusses the problem of finding that solution of (*) for which $|z|$ has the smallest positive value. He presents a continued fraction method which yields an upper bound (depending only on D) for the smallest attainable value of $|z|$.

A. L. Whiteman (Princeton, N.J.)

1541:

Stolt, B. Über einen verallgemeinerten Fermatschen Satz. Acta Arith. **5**, 267–276 (1959).

The author treats the Diophantine equation $Cx^2 + D = y^q$, where q is an odd prime, C and D are positive integers such that CD is square free, $CD \not\equiv 7 \pmod 8$, and q does not divide the class number $h(\sqrt{-CD})$. For certain values of C, D, and q the equation is shown to have no integer solutions. Let q and D be fixed, $D \neq 1, 3$. If $q \equiv 3 \pmod 4$ and $q > 3$, or if $q = 3$ and D is even, then there is at most one value of C, such that the above conditions are satisfied, for which the equation has solutions. For that value of C there is at most one solution in positive integers x, y. If $q = 3$ and D is odd, then as C runs through all admissible values there are at most two solutions in positive integers.

If $q \equiv 1 \pmod 4$, then D must be odd, and there is at most one solution for which C is odd and at most one solution for which $CD \equiv 6 \pmod 8$.

W. H. Mills (Berkeley, Calif.)

1542:

Nagell, T. Les points exceptionnels rationnels sur certaines cubiques du premier genre. Acta Arith. **5**, 333–357 (1959).

The principal result is theorem II: Let Ω be an algebraic number field with class-number 1 and suppose either that there are no units E, E_1 of Ω such that $1+E+E_1=0$ or that the only such units are cube roots of 1. Let a, b, c, be cube-free integers of Ω. Then there are no exceptional points defined over Ω on

$$ax^3+by^3+cz^3 = 0 \tag{1}$$

except when the equation (1) can be reduced to the shape

$$x'^3+y'^3+c'z'^3 = 0 \tag{2}$$

by writing $x=\varepsilon_1 x', y=\varepsilon_2 y', z=\varepsilon_3 z'$ with unit factors $\varepsilon_1, \varepsilon_2, \varepsilon_3$. If Ω is as above, all the exceptional points defined over Ω on the curve (2) can be specified completely. (An exceptional point is one which gives rise to only finitely many distinct points under the "chord and tangent process".) It is shown, further, that if Ω is either a quadratic field or a cubic field with negative discriminant and has class-number 1, then it satisfies the criterion of theorem II except in a finite number of cases which can be dealt with separately.

J. W. S. Cassels (Cambridge, England)

1543:

Cugiani, Marco. I domini P-adici e le forme binarie. Rend. Sem. Mat. Fis. Milano **29** (1959), 198–220. (English summary)

Expository paper, with special reference to the author's papers [Ist. Lombardo Accad. Sci. Lett. Rend. A **92** (1957/58), 307–320; Ann. Mat. Pura Appl. (4) **44** (1957), 1–22; Riv. Mat. Univ. Parma **8** (1957), 81–92; MR **21** #3380; **20** #1669, 7011].

J. W. S. Cassels (Cambridge, England)

1544:

Ward, Morgan. The vanishing of the homogeneous product sum of the roots of a cubic. Duke Math. J. **26** (1959), 553–562.

Let H_n denote the homogeneous product sum of x_1, x_2, x_3 of degree n:

$$H_n = \sum_{r_1+r_2+r_3=n} x_1^{r_1}x_2^{r_2}x_3^{r_3}.$$

The following results are proved. (1) If $n>1$, the diophantine equation $H_n=0$ has a non-trivial solution if and only if for this value of n there exists a trinomial $\phi(t)=t^{n+2}+pt+q$ with p and q relatively prime and $q\neq 0$ which has three integral roots. (2) If n is even the diophantine equation $H_n=0$ has only trivial solutions. (3) The diophantine equation $H_3=0$ has only trivial solutions.

The proof of the third result is by descent and makes use of properties of the field $R(\sqrt{-3})$. The author remarks that by the same method it can be proved that each of the equations $x_1^2x_2+\cdots+x_5x_3^2=0$ and $x^3+y^3+z^3=5xyz$ has only trivial solutions. The last result was obtained by Mordell [*Colloque sur la théorie des nombres, Bruxelles, 1955*, pp. 67–76, Thone, Liège, 1956; MR **17**, 1185] by a different method.

L. Carlitz (Durham, N.C.)

1545:

af Ekenstam, Adolf. ★Contributions to the theory of the Diophantine equation $Ax^n-By^n=C$. Inaugural dissertation, Univ. of Uppsala. Almqvist & Wiksells Boktryckeri AB, Uppsala, 1959. 63 pp.

Let A_i, B_i, C_i be positive integers, for $i=1, 2, \cdots, g$. If all the numbers $(A_i/B_i)^{1/n}$ define the same extension of the field K of rational numbers, the Diophantine equations $|A_ix^n-B_iy^n|=C_i$, for $i=1, 2, \cdots, g$, are said to belong to the same class. In the first chapter it is shown that at most one of a set of equations of the same class is solvable in relatively prime x and y, under the following hypotheses: $(A_iB_i, C_i)=1$, A_iB_i nth power free, $n\geqq 4$, and $\min_i(A_iB_i)$ larger than an explicitly given function of n and $\max_i(C_i)$. More precise results are obtained in the special cases $C_i=1$ for all i and $A_i=C_i=1$ for all i. In the second chapter the inequality $|Ax^n-By^n|\leqq C$ is considered, where A, B and C are fixed, and x and y are variable, integers of $K(\sqrt{-m})$, in the cases $n=m=3$ and $n=4$, $m=1$. It is shown that there is essentially at most one solution if $(A, B)=1$, $|A/B|^{2/n}$ is irrational, and $|AB|$ is larger than an explicit function of $|C|$. The first two chapters use the method of Siegel [Math. Ann. **114** (1937), 57–68] depending on identities between hypergeometric functions.

In Chapter 3 the following assertion of Tartakovskiĭ is proved: Let n be $\geqq 3$ and M a positive integer, not a square. If the equation $x^{2n}-My^{2n}=1$ has a solution $x=x_1, y=y_1$ in positive integers, then $x_1^n+y_1^n\sqrt{M}$ is the fundamental unit ε of $K(\sqrt{M})$, or its square, the latter occurring for only finitely many M. If $N(\varepsilon)=-1$, the equation has no solution with $y_1\neq 0$. The proof depends on a result of Siegel in the paper cited above.

W. J. LeVeque (Ann Arbor, Mich.)

1546:

Davenport, H. Some recent progress in analytic number theory. J. London Math. Soc. **35** (1960), 135–142.

The paper is a presidential address delivered to the London Mathematical Society on November 19, 1959. Among other items concerning publications, the author announces that the collected papers of G. H. Hardy will be published by the Oxford University Press and that the publication will probably extend to ten volumes.

The author then goes on to discuss some problems in analytic number theory. After giving a brief outline of the Hardy-Littlewood method he describes two variants of Waring's problem. First one can treat the representation of N as $a_1x_1^k+\cdots+a_nx_n^k$ and in particular the number of solutions of the equation $a_1x_1^k+\cdots+a_nx_n^k=0$ in integers $x_1, \cdots, x_n$; if k is even it is necessary to suppose that not all the coefficients are of the same sign. It can be proved by the Hardy-Littlewood method that the equation has infinitely many solutions if n is greater than a certain function of k. For $k=2$ the conclusion holds for $n\geqq 5$ as was proved by Meyer in 1883.

Secondly, one can treat the solvability of an inequality such as $|\lambda_1x_1^k+\cdots+\lambda_nx_n^k|<1$, where $\lambda_1, \cdots, \lambda_n$ are real numbers, not all of the same sign when k is even. It was proved by Davenport and Heilbronn in 1946 that the inequality has infinitely many solutions in integers provided $n\geqq 2^k+1$.

A generalization of the first problem above is that of solving an equation $F(x_1, \cdots, x_n)=0$, when F is a homogeneous polynomial of degree k with integral coefficients

(the first work on this problem, using the Hardy-Littlewood method, was that of Tartakovskiĭ in 1935). The corresponding generalization of the second problem is $|\Phi(x_1, \cdots, x_n)| < 1$, where Φ is a form of degree k with real coefficients. These problems are referred to as the 'equation problem' and the 'inequality problem', respectively. In either case it is assumed that the form represents zero for real $x_1, \cdots, x_n$ not all zero. The immediate new difficulty is caused by the non-additive nature of F or Φ. For the inequality problem with $k=2$ the author reports on results due to himself, Birch, Heilbronn and Ridout. It was proved that $|\Phi| < 1$ has an integral solution for $n \geq 21$; presumably $n \geq 5$ is the true solution. It is mentioned that "the case $k=3$ can also be solved, but that a new difficulty arises for $k>3$ which at present appears insuperable."

For the equation problem the first case of interest is $k=3$ since a quadratic form can be rationally transformed into a diagonal form. Davenport proved in 1957 that the cubic equation $F(x_1, \cdots, x_n)=0$ is solvable in integers (not all 0) if n is sufficiently large; D. J. Lewis had obtained a proof a little earlier. At about the same time Birch established the following more general result. Any system of equations

$$F_1(x_1, \cdots, x_n) = 0, \cdots; F_m(x_1, \cdots, x_n) = 0$$

of given odd degrees $k_1, \cdots, k_m$ is solvable in integers not all zero provided n is greater than a function of $k_1, \cdots, k_m$. For the case of a single cubic equation Davenport has proved that $n \geq 29$ suffices, but presumably $n \geq 10$ is the true condition.

Turning again to the work of Tartakovskiĭ it is noted that he proved that the 'general' equation, not necessarily homogeneous, of degree k in n variables is solvable, subject to the obvious necessary conditions, provided n is greater than a function of k. Birch has recently proved that the homogeneous equation $F(x_1, \cdots, x_n)=0$ is solvable in integers provided that the singular variety: $\partial F/\partial x_1 = \cdots = \partial F/\partial x_n = 0$ is of dimension $< n-(k-1)2^k$. In particular a non-singular cubic form in 17 or more variables represents zero.

In conclusion two problems concerning the singular variety of a primal are formulated.

L. Carlitz (Durham, N.C.)

1547:

Carlitz, L. Arithmetic properties of certain polynomial sequences. Bull. Amer. Math. Soc. **66** (1960), 202–204.

Consider the sequence of polynomials $\{u_n(x)\}$ that satisfy the recurrence

$$u_{n+1}(x) = (x+a(n))u_n(x)+b(n)u_{n-1}(x),$$

where $a(n)$, $b(n)$ are polynomials in n (and possibly some additional indeterminates) with integral coefficients. If it is assumed in addition that $u_0(x)=1$, $u_1(x)=a(0)$, $b(0)=0$, then the sequence $\{u_n(x)\}$ is uniquely determined. In the present paper it is proved that if $m \geq 1$, $r \geq 1$, then $u_n(x)$ satisfies the congruence

$$(1) \quad \sum_{s=0}^{r} (-1)^s \binom{r}{s} u_{n+sm}(x) u_m{}^{r-s}(x) \equiv 0 \pmod{m^{r_1}},$$

for all $n \geq 1$, where r_1 is the greatest integer $\leq \frac{1}{2}(r+1)$. Also it is shown that (1) implies

$$(2) \quad \sum_{s=0}^{r} (-1)^s \binom{r}{s} u_{n+sm}(x) u_{k+(r-s)m}(x) \equiv 0 \pmod{m^{r_1}},$$

for all $n \geq 0$, $k \geq 0$. For $k=0$, (2) reduces to a result proved by the author in a previous paper [Math. Z. **59** (1954), 474–483; MR **15**, 604].

A. L. Whiteman (Princeton, N.J.)

1548:

Grummich, Friedrich. Zur Zahlentheorie der Polynome. Ein Analogon zum Dirichlet'schen Satz. Math. Nachr. **17**, 330–357 (1959).

W. Specht in a series of papers [S.-B. Math. Nat. Kl. Bayer. Akad. Wiss. **1951**, 139–416; Math. Nachr. **7** (1952), 105–126, 127–150; Math. Z. **57** (1953), 291–335; MR **14**, 251, 278] has obtained asymptotic formulas for the number $I_m(x)$ of irreducible polynomials with integral coefficients $f(z)=z^m+a_1z^{m-1}+\cdots+a_m$ that satisfy $a_1{}^2+\cdots+a_m{}^2 \leq x^2$. In the present paper the author considers an analog of the Dirichlet theorem for polynomials in an arithmetic progression. Let $m \geq 2$ and let

$$B(z) = z^s+b_1z^{s-1}+\cdots+b_s \qquad (1 \leq s < m)$$
$$C(z) = c_1z^{s-1}+\cdots+c_s \qquad (1 \leq s < m)$$

be fixed polynomials with integral coefficients. Also let

$$f(z) = z^m+a_1z^{m-1}+\cdots+a_m,$$
$$g(z) = z^r+t_1z^{r-1}+\cdots+t_r$$

be polynomials with integral coefficients such that

$$(*) \qquad f(z) = B(z)g(z)+C(z).$$

For an arbitrary polynomial $\phi(z)=d_0z^n+\cdots+d_n$ put $N(\phi)=(d_0{}^2+\cdots+d_n{}^2)^{1/2}$. For $x>0$, let $\mathfrak{R}_{ms}(x)$ denote the set of polynomials $f(z)$ that satisfy (*) and such that $N(g)=(1+t_1{}^2+\cdots+t_r{}^2)^{1/2} \leq x$; also let $K_{ms}(x)$ denote the number of polynomials in the set. Let $I_{ms}(x)$ denote the number of irreducible polynomials in $\mathfrak{R}_{ms}(x)$ and $R_{ms}(x)$ the number of reducible polynomials. The object of the paper is to find asymptotic expressions for $I_{ms}(x)$ and $R_{ms}(x)$ as $x \to \infty$. Clearly for every m $K_{ms}(x)=\alpha_r x^r+O(\psi_r(x))$, $\alpha_r=\pi^{r/2}/\Gamma(\frac{1}{2}r+1)$, where $\psi_r(x)$ is the remainder term in the problem of lattice points in an r-dimensional sphere. The main results of the paper are the following. Put

$$\beta_n(B, C) = \sum_u (1+u^2+\cdots+u^{2n})^{-1/2},$$

where the summation is over all integral u such that $C(u)/B(u)$ is integral.

Theorem 1: For arbitrary $B(z)$,

$$R_{21}(x) = O(1),$$
$$I_{21}(x) = \alpha_1 x+O(1),$$
$$R_{31}(x) = \alpha_1\beta_1(B, C)x+O(1),$$
$$I_{31}(x) = \alpha_2 x^2-\alpha_1\beta_1(B, C)x+O(\psi_2(x)),$$
$$R_{32}(x) = O(1),$$
$$I_{32}(x) = \alpha_1 x+O(1),$$
$$R_{ms}(x) = \alpha_{r-1}\beta_{r-1}(B, C)x^{r-1}+O(x^{r-2}),$$
$$I_{ms}(x) = \alpha_r x^r-\alpha_{r-1}\beta_{r-1}(B, C)x^{r-1}+O(\psi_r(x)),$$

provided $2s \leq m-3$.

Theorem 2: For $B(z)=z^2+b_1z+b_2$, $\Delta=b_1{}^2-4b_2 \neq 0$,

$$R_{42}(x) = \alpha_1\beta_1(B, C)x+O(\log x),$$
$$R_{m2}(x) = \alpha_{r-1}\beta_{r-1}(B, C)x^{r-1}+O(x^{r-2}) \quad (m \geq 5),$$
$$I_{m2}(x) = \alpha_r x^r-\alpha_{r-1}\beta_{r-1}(B, C)x^{r-1}+O(\psi_r(x)) \quad (m \geq 4).$$

Theorem 3: For $B(z)=(z-b)^p$, $2s \leqq m+3$,

$$R_{m,m-1}(x) = O(1),$$
$$I_{m,m-1}(x) = \alpha_1 x + O(1),$$
$$R_{m,m-2}(x) = \alpha_1\beta_1(B,C)x + O(x^{1/2}),$$
$$I_{m,m-2}(x) = \alpha_2 x^2 - \alpha_1\beta_1(B,C)x + O(\psi_2(x)),$$
$$R_{m,m-3}(x) = \alpha_2\beta_2(B,C)x^2 + \beta'_{m-3}x\log x + O(x),$$
$$I_{m,m-3}(x) = \alpha_3 x^3 - \alpha_2\beta_2(B,C)x^2 - \beta'_{m-3}x\log x + O(\psi_3(x)),$$

where β'_{m-3} is a constant depending on b and C,

$$R_{ms}(x) = \alpha_{r-1}\beta_{r-1}(B,C)x^{r-1} + O(x^{r-2}),$$
$$I_{ms}(x) = \alpha_r x^r - \alpha_{r-1}\beta_{r-1}(B,C)x^{r-1} + O(\psi_r(x))$$

and $s \leqq m-4$ in the last two formulas.

The author remarks that unless s is small as compared with m, the results may depend on properties of $B(z)$. This motivates Theorems 2 and 3. Thus for example the order of $R_{52}(x)$ depends on whether the discriminant of $B(z)$ does or does not vanish.

L. Carlitz (Durham, N.C.)

1549:

Carlitz, L. Some finite summation formulas of arithmetic character. Publ. Math. Debrecen **6** (1959), 262–268.

Let $f_1(x), f_2(x), \cdots, f_n(x)$ be functions of x such that for any positive integer k there exists the relation

$$(*) \qquad \sum_{r=0}^{k-1} f_i(x+r/k) = C_i^{(k)} f_i(kx) \quad (i = 1, \cdots, n),$$

where $C_i^{(k)}$ is independent of x. Three theorems are proved in this paper of which the following is theorem 1. Let $n \geqq 1$ and let $a_1, \cdots, a_n$ be positive integers that are relatively prime in pairs and put $A = a_1 a_2 \cdots a_n$. Let $f_1(x), \cdots, f_n(x)$ be functions of x of period 1 that satisfy (*). Then if k is an arbitrary positive integer, we have

$$\sum_{r=0}^{kA-1} f_1\left(\frac{r}{a_1 k}\right) f_2\left(\frac{r}{a_2 k}\right) \cdots f_n\left(\frac{r}{a_n k}\right) =$$
$$C_1^{(a_1)} C_2^{(a_2)} \cdots C_n^{(a_n)} \sum_{r=0}^{k-1} f_1\left(\frac{r}{k}\right) f_2\left(\frac{r}{k}\right) \cdots f_n\left(\frac{r}{k}\right).$$

The proof is very similar to the proof of a related theorem by Mordell [J. London Math. Soc. **33** (1958), 371–375; MR **20** #6488] which involves integrals in place of finite sums. Let $B_m(x)$ be the Bernoulli polynomial of degree m, and let $\bar{B}_m(x)$ be the corresponding Bernoulli function; let $\zeta(s, x)$ denote the Hurwitz zeta-function. Both $B_m(x)$ and $\zeta(s, x)$ satisfy relations of the form (*). Theorems 2 and 3 are applications of theorem 1 and are derived by putting $n=2, f(x)=\bar{B}_m(x)$ and $n=2, f(x)=\zeta(s, x)$ respectively.

A. L. Whiteman (Princeton, N.J.)

1550:

Yamamoto, Koichi. On the Kummer-Hilbert reciprocity law. Mem. Fac. Sci. Kyushu Univ. Ser. A. **13** (1959), 85–95.

Let p be an odd prime, Q the field of rational p-adic numbers, ζ a primitive pth root of unity, $k=Q(\zeta)$, μ and ν principal units of k, and $M^p=\mu$. A necessary and sufficient condition is given for ν to be a norm of the kummerian extension $k(M)/k$. {Lemmas 5 and 5′ are false. This appears to invalidate the rest of the argument.}

W. H. Mills (Berkeley, Calif.)

1551:

Shiratani, Katsumi. On potent functions in local number fields. Mem. Fac. Sci. Kyushu Univ. Ser. A. **13** (1959), 126–135.

Let p be a prime, R the rational p-adic field, $\bar{R}$ the algebraic closure of R, $\bar{T}$ the maximal unramified field in $\bar{R}$, $O_{\bar{T}}$ the ring of integers of $\bar{T}$, and P the Frobenius automorphism of $\bar{T}/R$. Then, for $\alpha \in O_{\bar{T}}$, $E(\alpha, x) = \exp(-\sum_{i=0}^{\infty} \alpha^{p^i} p^{-i} x^{p^i})$ and $F(\alpha, x) = \exp(-\sum_{i=0}^{\infty} \alpha^{p^i} i^{-1} x^{i})$ define power series in x. These power series played an important role in recent investigations of reciprocity laws [I. R. Šafarevič, Mat. Sb. (N.S.) **26 (68)** (1950), 113–146; B. Dwork, Abh. Math. Sem. Univ. Hamburg **22** (1958), 180–190; MR **11**, 230; **20** #4541]. The author investigates the convergence of these two power series, and uses his results to obtain explicit representations of p^n-primary numbers in a suitable field k.

W. H. Mills (Berkeley, Calif.)

1552:

Tatuzawa, Tikao. On the Hecke-Landau L-series. Nagoya Math. J. **16** (1960), 11–20.

Let k be an algebraic number field of degree $n = r_1 + 2r_2$ with discriminant d. If f is an integral ideal, let $\bar{f} = f f_\infty$ where f_∞ is a product of infinite primes, say $f_\infty = p_\infty^{(1)} p_\infty^{(2)} \cdots p_\infty^{(q)}$ $(0 \leqq q \leqq r_1)$. The generalized strahl class S is defined by the conditions $\alpha \equiv 1 \pmod f$, $\alpha^{(i)} > 0$ $(i = 1, \cdots, q)$. In this way, the author introduces a generalized character $\chi \bmod \bar{f}$ as a character on the Abelian group A/S where A is the group of all ideals prime to f. If the conductor of χ is $\bar{f}$, χ is called primitive. For a primitive character χ let

$$L(s, \chi) = \sum_{\mathfrak{a}} \chi(\mathfrak{a}) N(\mathfrak{a})^{-s},$$

$$\varphi(s, \chi) = \frac{(2\pi)^{r_2}}{\sqrt{d}} A^{s/2}(\chi) \Gamma(s, \chi) L(s, \chi),$$

$$\psi(s, \chi) = \frac{(2\pi)^{r_2}}{\sqrt{d}} A^{s/2}(\chi) \sum_{\mathfrak{a}} \chi(\mathfrak{a}) \Gamma(s, \chi, \mathfrak{a}) N(\mathfrak{a})^{-s},$$

where $A(\chi) = \pi^{-n} d N(f)$, $\Gamma(s, \chi)$ is a product of gamma factors and $\Gamma(s, \chi, \mathfrak{a})$ is a multiple "incomplete" gamma function.

The main theorem the author proves is the following generalization of Hecke's formula:

$$\varphi(s, \chi) = -\frac{2^{r_1+r_2}\pi^{r_2} R h}{w\sqrt{d}} \frac{E(\chi)}{s(1-s)} + \psi(s, \chi) + I(\chi)\psi(1-s, \bar{\chi}),$$

where R, w, h, $E(\chi)$ have their usual significance and $I(\chi)$ is a certain normalized Gaussian sum. In particular, there follows a functional equation for $\varphi(s, \chi)$, namely $\varphi(s, \chi) = I(\chi)\varphi(1-s, \bar{\chi})$, as well as the familiar formula

$$\pi^{-s/2}\Gamma(s/2)\zeta(s) = \frac{1}{s(s-1)} + \pi^{-s/2} \sum n^{-s} \int_{\pi n^2}^{\infty} e^{-z} z^{s/2} \frac{dz}{z}$$
$$+ \pi^{-(1-s)/2} \sum n^{-1+s} \int_{\pi n^2}^{\infty} e^{-z} z^{(1-s)/2} \frac{dz}{z}$$

for the Riemann zeta function.

R. Ayoub (University Park, Pa.)

1553:

van Lint, J. H. Linear relations for certain modular forms. Math. Nachr. **20** (1959), 123–126.

Let f be a modular form of step N, let r be half of an odd integer, and denote by Γ_0 the subgroup of the modular

group $\Gamma(1)$ that is generated by $\begin{pmatrix}1&2\\0&1\end{pmatrix}$ and $\begin{pmatrix}0&-1\\1&0\end{pmatrix}$. Also, denote by η and θ the Dedekind function and $\sum_n e^{\pi i n^2}$, respectively. In his thesis [Utrecht, 1957; MR **19**, 839] the author proved the following theorem: The only integral modular forms $\in\{\Gamma_\theta, -r, v\}$ for which linear relations of type $F(p\tau)+\alpha\sum_{l \bmod p} F((\tau+\lambda l)/p)=0$ exist for all p with $(p, \frac{1}{2}N)=1$ are η, η^3, θ, $(\eta\theta)^{1/2}$, $(\eta^3\theta^{-1})^{1/2}$, $(\eta\theta^5)^{1/2}$, $(\eta\theta)^{3/2}$ and $\eta^4\theta^{-1}$. Continuing the study of the same topic, the author now proves the following. The only modular forms $\in\{\Gamma_\theta, -r, v\}$, where v is not a multiplier system for $\Gamma(1)$, for which singular Hecke operators can be defined for all p with $(p, \frac{1}{2}N)=1$, are those listed above, except η and η^3. For the case of linear relations of order 1, the following is shown: If a linear relation of order 1 exists for a modular form $f\in\{\Gamma_\theta, -r, v\}$, then v is the restriction to Γ_θ of a multiplier system of $\Gamma(1)$ and the relation has the form $f|T[K, \Lambda]=0$, where $\Lambda=\{\Gamma(1), -r, v\}$, unless f is already a modular form of Λ.

E. Grosswald (Princeton, N.J.)

1554:

Newman, M. Weighted restricted partitions. Acta Arith. **5**, 371–380 (1959).

From the Dedekind eta-function $\eta(\tau)=x^{1/24}\prod(1-x^n)$ $(x=\exp 2\pi i\tau)$, the author defines $B(\tau)=\eta(q\tau)/\eta(\tau)=x^{(q-1)/24}\varphi$, whence $\varphi=\prod(1-x^{nq})/(1-x^n)$. Here $B(\tau)$ is an entire modular function of level q, and we further introduce $g(\tau)=B^p(\tau)B^q(p\tau)$ for p, q different primes, and $p>3$. The author shows that $g(\tau)$ is an entire modular function on $\Gamma_0(pq)$ if $(r+sp)(q-1)/24$ and $(r+s)/2$ are integers. He also shows that $G(\tau)=\sum g(R\tau)$, summed over the coset operations R of $\Gamma_0(q)/\Gamma_0(pq)$, is an entire modular function on $\Gamma_0(q)$.

With $r=-sp$, the main theorem emerges: If $s\geqq 0$, $G(\tau)$ has a pole of order $M=[|s|(p^2-1)(q-1)/24p]$ at $i\infty$ and is pole-free at 0, while if $s<0$, then $G(\tau)$ is pole-free at $i\infty$ and has a pole of order M at 0. If $q=2, 3, 5, 7, 13$, then $\Gamma_0(q)$ has genus 0 and, easily, $G(\tau)$ is a polynomial of degree M in $B(\tau)^{-24/(q-1)}$. This leads to identities among coefficients of $\varphi^s=\sum q_s(n)x^n$ of type

$$pq_1(np+\nu)+(-1)^r q_1(n/p) = 2^{(p-1)/2}q_p(n),$$

where $\nu=(p^2-1)/24$, $q=2$ and $5\leqq p\leqq 23$.

Incidentally, the coefficients $q_s(n)$ are weighted restricted partition functions. *H. Cohn* (Tucson, Ariz.)

1555:

Staś, W. Über eine Abschätzung des Restgliedes im Primzahlsatz. Acta Arith. **5**, 427–434 (1959).

Let $\psi(x)=\sum_{n\leqq x}\Lambda(n)=\sum_{p^m\leqq x}\log p$, $R(x)=\psi(x)-x$, where n and m run over positive integers, and p over primes. Let $\zeta(s)$ be the Riemann zeta-function. The main result of this paper is: If $\zeta(\rho_0)=0$, $\rho_0=\beta_0+i\gamma_0$ $(\frac{1}{2}\leqq\beta_0<1)$, and $T>\max(c, \exp\exp(2|\rho_0|))$, where c is an explicit numerical constant, then

$$\max_{x}|R(x)| > T^{\beta_0}\exp\left(-8\frac{\log T}{\log\log T}\right),$$

where the max relates to the range

$$T\exp\left(\frac{-\log T\log\log\log T}{(\log\log T)^2}\right)-1\leqq x\leqq T.$$

A result of this kind for the range $1\leqq x\leqq T$ had previously been obtained by Turán with the aid of his new method in analysis [*Eine neue Methode in der Analysis und deren Anwendungen*, Akadémiai Kiadó, Budapest, 1953; MR **15**, 688; p. 113]. The main point of the new result is the closer localization of x. The proof is based on a study of the integral

$$I_k=\frac{1}{2\pi i}\int_{(2)}\left(e^{A\omega}\frac{e^{B\omega}-e^{-B\omega}}{2B\omega}\right)^k F(\omega)d\omega,$$

$$F(s) = -\zeta(s)-\frac{\zeta'}{\zeta}(s),$$

where $A=\log\log T$, $B=1/A$, k is a positive integer (roughly of order $(\log T)/(\log\log T)$). By expressing I_k in terms of $\Lambda(n)$ the author first obtains an upper estimate of $|I_k|$ in terms of $\max|R(x)|$ $(e^{(A-B)k}-1\leqq x\leqq e^{(A+B)k})$. A lower estimate is then obtained, for a suitable k, by expressing I_k in terms of a sum over the complex zeros $\rho=\beta+i\gamma$ of $\zeta(s)$ and applying one of Turán's fundamental theorems to a finite part $|\gamma|<T_1$ of this sum, where T_1 is about $(\log T)/(\log\log T)^4$. Combination of the two estimates gives the desired result.

A. E. Ingham (Cambridge, England)

1556:

Knapowski, S. On the zeros of certain Dirichlet polynomials. Nederl. Akad. Wetensch. Proc. Ser. A **63**= Indag. Math. **22** (1960), 54–58.

The author considers a Dirichlet series $f(s)=\sum_1^\infty a_n n^{-s}$ with some of the properties of the Riemann zeta function, viz.: (1) $a_1=1$, $a_n=O(1)$; (2) $f(s)$ is regular and $\neq 0$ on the line $s=1+it$, save at $s=1$ where it has either a pole or a zero; (3) $(f(s))^{-1}=\sum_1^\infty b_n n^{-s}$, $f'(s)/f(s)=\sum_1^\infty r_n n^{-s}$ with $b_n=O(1)$, $r_n=O(\log n)$; (4) either $r_n\leqq 0$ for all n or $r_n\geqq 0$ for all n. He proves that, if N is sufficiently large, the partial sum $\sum_1^N a_n n^{-s}$ has at least one zero in a small rectangle around the point $s=1$, viz., the rectangle described by

$$|\sigma-1|\leqq 2\log\log N/\log N, \quad |t|\leqq 3\log\log N(\log N)^{-1/2}.$$

{The author's statement that the theorem applies to the Dedekind zeta functions is incorrect, for there the a_n are not always bounded.} *N. G. de Bruijn* (Eindhoven)

1557:

Besicovitch, A. S. Rational polygons. Mathematika **6** (1959), 98.

The author proves that there is a right-angled triangle whose sides, all having rational lengths, are arbitrarily near to those of any given right-angled triangle. He proves, further, that there is a parallelogram whose sides and diagonals, all having rational lengths, are arbitrarily near to those of any given parallelogram. (The spelling "paralellogram" is unusual.)

H. S. M. Coxeter (Toronto)

1558:

Connell, Ian G. Some properties of Beatty sequences. II. Canad. Math. Bull. **3** (1960), 17–22.

The complementary sequences $\{u_n\}$ and $\{v_n\}$ defined by $u_n=[n(1+1/\alpha)]$, $v_n=[n(1+\alpha)]$ $(n=1, 2, 3, \cdots)$, where square brackets denote the greatest integer function and α is any positive irrational, are called Beatty sequences of argument α. Some properties of these sequences were discussed by the author in part I [same Bull. **2** (1959),

190–197; MR **21** #7805]. This paper is devoted to the study of further properties which are connected with simple continued fractions. An algorithm to determine the continued fraction expansion of α is obtained.

M. Dresher (Pacific Palisades, Calif.)

1559:

Berg, Lothar. Massbestimmung linearer Punktmengen. II. Math. Nachr. **17**, 211–218 (1959).

This paper is a continuation of two earlier publications of the author concerning the metrical properties of a certain class of expansions of real numbers [Math. Nachr. **14** (1955), 263–285; **16** (1957), 195–205; MR **18**, 645; **20** #94]. The author first generalizes his "Homogenitätssatz" (Satz 1, p. 270, in the first paper). Then he studies more closely the measure of the set M already introduced in the two former papers (cf. Satz 3, p. 276, in the first paper, and Satz 1, p. 196, in the second). If this measure is zero, then an analogue of the Bernstein-Knopp theorem from the theory of continued fractions holds for the expansions considered. In all the former special cases treated by the author, this set turned out to be homogeneous, hence has measure zero or one. Now it is shown, among other results, that this need not be the case in general.

J. Popken (Amsterdam)

1560:

Bergström, Harald. The fundamental theorem of Roth. Nordisk Mat. Tidskr. **7** (1959), 57–72, 96. (Swedish. English summary)

An exposition of the original theorem of Roth [Mathematika **2** (1955), 1–20; corrigendum, 168; MR **17**, 242], somewhat simplified by the systematic use of vectors.

W. J. LeVeque (Ann Arbor, Mich.)

1561a:

Ehrhart, Eugène. Nombre de solutions de l'équation et de l'inéquation diophantiennes linéaires à trois inconnues. C. R. Acad. Sci. Paris **248** (1959), 620–623.

1561b:

Ehrhart, Eugène. Nombre de solutions de l'équation diophantienne linéaire à trois inconnues. C. R. Acad. Sci. Paris **248** (1959), 758–761.

1561c:

Ehrhart, Eugène. Sur les équations diophantiennes linéaires à plus de trois inconnues. C. R. Acad. Sci. Paris **248** (1959), 896–899.

1561d:

Ehrhart, Eugène. Nombre de solutions d'un système d'inéquations diophantiennes linéaires à deux ou trois inconnues, à trois ou quatre si on lui adjoint une équation. C. R. Acad. Sci. Paris **248** (1959), 1096–1099.

1561e:

Ehrhart, Eugène. Sur les polyèdres entiers à n dimensions. C. R. Acad. Sci. Paris **248** (1959), 1281–1284.

1561f:

Ehrhart, Eugène. Sur les polyèdres rationnels à n dimensions. C. R. Acad. Sci. Paris **250** (1960), 272–274.

Further instalments of the author's study "Corps convexes, réseaux entiers, équations et inéquations diophantiennes linéaires" as distilled into Comptes Rendus notes since 1955 [e.g., **246** (1958), 3142–3145; MR **20** #5189]. These give a large number of theorems and conjectures, said to be supported by numerical evidence, about the relations between the numbers of points in and on the surface of homothetic polyhedra in several dimensions. The theorems seem to be pretty straightforward; for example, relating the number of positive integer solutions (X, Y, Z) of $aX+bY+cZ=n$, where a, b, c are given positive integers and two values of n are considered which are congruent modulo each of a, b, c. The conjectures may be more interesting but are too circumstantial to quote here.

J. W. S. Cassels (Cambridge, England)

1562a:

Ehrhart, Eugène. Systèmes diophantiens linéaires. C. R. Acad. Sci. Paris **250** (1960), 643–645.

1562b:

Ehrhart, Eugène. Sur les polyèdres rationnels et les systèmes diophantiens linéaires. C. R. Acad. Sci. Paris **250** (1960), 959–961.

These are notes 25 and 26 of a sequence [see preceding review] mainly about points in and on the surface of polyhedra in a space of V dimensions. The second enunciates four conjectures all of which have been verified for $V=1$ and two also for $V=2$. They have also been verified in some special cases. The first note contains numerical examples which are stated to be in accordance with these and other conjectures of the author.

J. W. S. Cassels (Cambridge, England)

1563:

Ehrhart, Eugène. Sur les polyèdres entiers ou rationnels homothétiques. C. R. Acad. Sci. Paris **250** (1960), 1428–1430.

This is, according to the author, the last of a series of nearly thirty notes with the general title "Corps convexes, réseaux entiers, équations et inéquations diophantiennes linéaires" [cf. two preceding reviews]. In this note he tidies up points left from previous instalments and supplies a list of errata.

J. W. S. Cassels (Cambridge, England)

1564:

Mordell, L. J. Tschebotareff's theorem on the product of non-homogeneous linear forms. II. J. London Math. Soc. **35** (1960), 91–97.

[For part I see Vierteljschr. Naturf. Ges. Zürich **85** Beiblatt (1940), 47–50; MR **2**, 350.] Let $L_1, L_2, \cdots, L_n$ be n linear forms in the n variables $(u)=(u_1, u_2, \cdots, u_n)$ with real coefficients a_{rs} such that $\Delta=|\det a_{rs}|>0$. Let $c=(c_1, c_2, \cdots, c_n)$ be any real numbers, let M denote the lower bound for all sets of integers (u) of $\prod_{r=1}^{n}|L_r+c_r|$, and let T_n be $2^{n/2}M/\Delta$. Minkowski conjectured that $T_n \leqq 2^{-n/2}$ and this has been proved for $n=2, 3, 4$. Tschebotareff proved that $T_n \leqq 1$ and the author and A. C. Woods improved this result. Davenport showed that $T_n < \gamma_n^{-1}$, where $\gamma_n > 1$ and $\gamma_n \to 2e-1$ as $n \to \infty$. In this paper the author proves

$$T_n \leqq [4-2(2-(3/4)\cdot 2^{1/2})^n - 2^{-n/2}]^{-1}.$$

This improves Woods' result but is not as good as Davenport's for large n.

B. W. Jones (Boulder, Colo.)

FIELDS

See also 1571a-b.

1565:

Rédei, Ladislaus. Ein spezieller Diskriminantensatz über Polynome. Acta Sci. Math. Szeged **20** (1959), 234–237.

Let $F(x)=x^4+px^2+qx+r$ be a quartic polynomial over a field K whose characteristic is neither 2 nor 3, and let

$$G(y) = y^3-2py^2+(p^2-4r)y+q^2$$

be the cubic resolvent of F. The following problem is posed and solved: to find all polynomials of the form

$$H(x, y) = ax^2y^2+dx^2y+bxy^2+gx^2+exy+cy^2+hx+fy+k$$

with coefficients in a suitable extension field such that the discriminants of H with respect to y and x are equal to F and G respectively, that is

$$H_y = (dx^2+ex+f)^2-4(ax^2+bx+c)(gx^2+hx+k) = F(x),$$

$$H_x = (by^2+ey+h)^2-4(ay^2+dy+g)(cy^2+fy+k) = G(y).$$

The author explicitly determines all such polynomials $H(x, y)$. Exactly one of them is of degree 3.

W. Lederman (Manchester)

1566:

Demuškin, S. P.; Šafarevič, I. R. The imbedding problem for local fields. Izv. Akad. Nauk SSSR. Ser. Mat. **23** (1959), 823–840. (Russian)

Authors' introduction, somewhat modified: "In this paper we investigate conditions under which a given normal field-extension k/Ω with Galois group F may be embedded in a larger field-extension K/Ω with Galois group G so that a given epimorphism ϕ: $G\to F$ is realized as the natural mapping of the Galois group of a field onto that of a subfield. We consider only the case when the kernel A of the epimorphism ϕ is abelian and when k contains all the roots of unity whose order is the period of A (i.e., the least common multiple of the orders of the elements of A).

"The problem consists in finding invariants of the field k, the group G and the epimorphism ϕ on which the solvability of the problem of embedding depends. Such invariants we call obstructions. A set of conditions necessary for the solvability of the problem of embedding was obtained by D. K. Faddeev [B. N. Delone and D. K. Faddeev, Mat. Sb. (N.S.) **15** (**57**) (1944), 243–276; MR **6**, 200] and rediscovered by H. Hasse [Math. Nachr. **1** (1948), 40–61; MR **10**, 426]. We call these conditions the first obstruction. It consists of a set of elements of the groups ${}^2H(F_x, k^*)$ where $x\in \mathrm{Hom}(A, k^*)$ and F_x is the set of elements of F which leave x invariant when F acts on $\mathrm{Hom}(A, k^*)$ in the natural way. If the first obstruction vanishes we shall say that the field extension k/Ω is compatible with the given problem of embedding. In general compatibility is not sufficient for solvability.

"By a theorem of Kochendörffer [Math. Nachr. **10** (1953), 75–84; MR **15**, 282] it is enough to consider the problem of embedding for p-groups. Then we may suppose that A contains a subgroup A_1 of index p which is normal in G. Then A/A_1 is in the centre of G/A_1. If the field extension k/Ω is compatible with the original embedding problem, then it is easy to see that the problem of embedding in a field of Galois group G/A_1 is actually soluble. The solution is a field $k(\mu^{1/p})$, where $\mu\in k$ is determined only up to multiplication by an element m of Ω. We investigate when μ may be chosen so that the extension $k(\mu^{1/p})/\Omega$ is compatible with the problem of embedding in an extension K/Ω with group G. For given μ the first obstruction consists of a set of cyclic algebras of order p. When μ is replaced by $m\mu$, these algebras behave in a very simple way.

"Finally, we show that the vanishing of the first obstruction is both necessary and sufficient for the solvability of the embedding problem when Ω is a finite extension of the field of p-adic numbers. This follows from the general theory just discussed and from general properties of the cohomology groups of such fields."

J. W. S. Cassels (Cambridge, England)

ALGEBRAIC GEOMETRY

1567:

Winger, R. M. On a family of generalized conchoids. Amer. Math. Monthly **66** (1959), 843–849.

L'object de cet article est, par l'emploi des coordonnées isotropes, d'étudier les courbes définies en coordonnées polaires par $\rho=a\sec p\theta/q+k$, où a et k sont des nombres réels, tandis que p et q sont des nombres entiers premiers entre eux.

F. Şemin (Istanbul)

1568:

Rosina, B. A. Sugli spazi lineari contenuti in un'ipersuperficie con un punto multiplo. Rend. Sem. Mat. Univ. Padova **28** (1958), 290–321.

Giovandosi di risultati di U. Morin ed A. Predonzan l'A. studia i sistemi di spazi S_k contenuti in una V^n_{r-1} algebrica di S_r dotata di un punto O $(n-s)$-plo. Se V^n_{r-1} ha equazione: $\varphi_n+x_0\varphi_{n-1}+\cdots+x_0^s\varphi_{n-s}=0$, ove φ_i è una forma di grado i nelle sole variabili $x_1, \cdots, x_r$, e se Φ denota la varietà $x_0=\varphi_n=\varphi_{n-1}=\cdots=\varphi_{n-s}=0$, la V^n_{r-1} possiede degli S_k passanti per O o situati in S_{k+1} passanti per O ed appartenenti a V^n_{r-1}, se Φ contiene degli S_{k-1} o rispettivamente degli S_k; l'esistenza di altri S_k su V^n_{r-1} equivale all'esistenza di $V^{(n-1)(n-2)\cdots(n-s)}_{k-s}$ di S_k contenute in Φ.

D. Gallarati (Genoa)

1569:

Godeaux, Lucien. Note sur les surfaces de genres zéro possédant un réseau irréductible de courbes bicanoniques. III. Acad. Roy. Belg. Bull. Cl. Sci. (5) **45** (1959), 188–196.

The author continues the work of earlier papers [same Bull. (5) **45** (1959), 52–68; MR **21** #6369] and completes the account of the possible types of surfaces in S_8 of order 18 and hyperplane section genus 13 which are tricanonical models of the surfaces with the required properties.

T. G. Room (Sydney)

1570:

Herrmann, Manfred. Über birationale Berührungstransformationen zweiter Ordnung. Wiss. Z. Martin-Luther-Univ. Halle-Wittenberg. Math.-Nat. Reihe **8** (1958/59), 375–386.

The author discusses systems of conics with the "Ω-property", namely, that the system is rational of freedom 2, that exactly one member touches a generic line at an assigned generic point of itself, and that of the common points of two neighbouring curves of the system all but one is fixed. To calculate the various characteristic members involved, he discusses in turn the possible ∞^1-systems of conics which satisfy the conditions that they touch a line, pass through r ($r=2, 1, 0$) points, and touch a given c^n ($n \geqq 4-r$) at $3-r$ points, the various cases of coincidence among these points being considered.

T. G. Room (Sydney)

1571a:

Li, Syan-pin [Li, Shiang-ping]. Characteristic properties of algebraic surfaces and determination of the factors of polynomials. Acta Math. Sinica **7** (1957), 492–512. (Chinese. Russian summary)

1571b:

Li, Shiang-ping. Characteristic properties of algebraic surfaces and determination of the factors of polynomials. Sci. Sinica **7** (1958), 1001–1026.

[The second article is a translation of the first.]

The author tries to find a necessary and sufficient condition for a surface in a complex Euclidean space U^n to be algebraic. Here by a surface S is meant a proper closed subset of U^n such that any complex straight line L in U^n either is contained in S, does not intersect S at all, or intersects S at a fixed finite number m of points, where the multiplicity of intersection is defined topologically. Let $a, b \in U^n - S$ and let $L(a, b)$ be the straight line joining a and b. Let $q_i = (\lambda_i a + b)/(\lambda_i + 1)$ $(i = 1, \cdots, m)$ be points of intersection of $L(a, b)$ and S. Then the ratio product S_{ab} is defined to be $(\lambda_1 \cdots \lambda_m)^2$. If S is an algebraic hypersurface, then $S_{ab}S_{bc}\cdots S_{ka} = 1$ holds for arbitrary points $a, b, \cdots, k$ in $U^n - S$ (Theorem 3). The author claims that, conversely, if the above condition holds then S is algebraic (Theorem 4). As an application of these results the author gives an explicit method of determination of the types of factors of various degrees of polynomials and the condition for the reducibility of polynomials. {The reviewer cannot follow the proof of Theorem 4 at p. 1016.}

Y. Kawada (Tokyo)

1572:

★Séminaire C. Chevalley; 2e année: 1958. Anneaux de Chow et applications. Secrétariat mathématique, 11 rue Pierre Curie, Paris, 1958. iii+134 pp. (mimeographed)

Exposé 1, par J. P. Serre, sur "Les espaces fibrés algébriques". L'auteur définit d'abord les revêtements non-ramifiés, étudie diverses opérations sur ceux-ci, et s'occupe en particulier des revêtements galoisiens. Un système fibré (algébrique) est un triplet (G, P, X) (P, X espaces algébriques, G groupe algébrique opérant à droite sur P), muni d'une projection $p: P \to X$ telle que $p(x \cdot g) = p(x)$ pour $x \in P$ et $g \in G$. Un système fibré (G, P, X) est appelé un espace fibré principal de base X et de groupe G s'il est "localement isotrivial", c'est à dire s'il existe un revêtement non-ramifié $X' \to X$ tel que l'image réciproque de P soit localement triviale. Lien avec les cocycles. Critères d'isotrivialité locale. Exemples. Opérations sur les espaces fibrés principaux. Critères de trivialité locale. Classification des espaces fibrés principaux sur certains groupes particuliers. Comparison avec les espaces fibrés analytiques.

Exposés 2 et 3, par C. Chevalley, sur "Les classes d'équivalence rationnelle". On y montre, avec grand soin, que, pour l'équivalence rationnelle, les classes de cycles sur une variété quasi-projective non-singulière forment un anneau ("l'anneau de Chow"). L'existence d'un cycle X' rationnellement équivalent à un cycle donné X et jouissant de "bonnes propriétés d'intersection" est démontrée par applications répétées de la méthode des cônes (l'application simple, comme dans P. Samuel, *Rational equivalence of arbitrary cycles* [Amer. J. Math. **78** (1956), 383–400; MR **20** #2343], étant insuffisante). Fonctorialité de l'anneau de Chow.

Exposé 4, par A. Grothendieck, sur "Quelques propriétés fondamentales en théorie des intersections". Formulaire axiomatique d'une théorie globale des intersections. Classes de Poincaré, et classes caractéristiques des fibrés de rang 1. Classes de Chern, et fibrés en drapeaux. Propriétés particulières (d'exactitude et d'homotopie) des anneaux de classes de cycles. Applications.

Exposé 5, par A. Grothendieck, sur "La torsion homologique et les sections rationnelles". Étude des classes de cycles sur un espace fibré. Construction de sections d'espaces fibrés au moyen de représentations linéaires. Existence de sections rationnelles d'espaces fibrés sous certaines hypothèses. Notion de torsion d'un groupe algébrique, et structure des groupes algébriques semi simples sans torsion. Relations avec la torsion homologique dans le cas classique.

P. Samuel (Clermont-Ferrand)

1573:

Chow, Wei-Liang. On the connectedness theorem in algebraic geometry. Amer. J. Math. **81** (1959), 1033–1074.

Let $(\mathfrak{R}, \mathfrak{p})$ be a pair of a regular local ring $\mathfrak{R}$ and its maximal ideal $\mathfrak{p}$. Let K be the quotient field of $\mathfrak{R}$, $\overline{\mathfrak{R}} = \mathfrak{R}/\mathfrak{p}$; if a [resp. $f(X)$] is an element of $\mathfrak{R}$ [resp. $\mathfrak{R}[X]$], denote by $\bar{a}$ [resp. $\bar{f}(X)$] the image of a [resp. $f(X)$] under the canonical mapping $\mathfrak{R} \to \overline{\mathfrak{R}}$. If y is a point in a projective space S^m, η is called a specialization of y at $\mathfrak{R}$, if, for every form $f(Y)$ in $\mathfrak{R}[Y]$, the relation $f(y) = 0$ implies $\bar{f}(\eta) = 0$. If Z is a positive cycle in a projective space, a specialization of Z at $\mathfrak{R}$ is defined in terms of its Chow-point in the obvious manner. The set of all specializations of y [resp. Z] is denoted by $\bar{y}(\mathfrak{R})$ [resp. $\bar{Z}(\mathfrak{R})$]. The union of the supports of cycles in $\bar{Z}(\mathfrak{R})$ is denoted by $|\bar{Z}(\mathfrak{R})|$. It is easy to see that $\bar{y}(\mathfrak{R})$, and hence $|\bar{Z}(\mathfrak{R})|$, are $\overline{\mathfrak{R}}$-closed in their ambient projective spaces.

From now on, let us assume that y is rational over K. Let $\mathscr{V}(\mathfrak{R})$ be the set of valuation-rings in K dominating $\mathfrak{R}$; then the map $M \to \bar{y}(M)$, $M \in \mathscr{V}(\mathfrak{R})$, is shown to be a surjective mapping of $\mathscr{V}(\mathfrak{R})$ to $\bar{y}(\mathfrak{R})$, and thus a topology can be introduced in $\mathscr{V}(\mathfrak{R})$ by means of the Zariski-topology on $\bar{y}(\mathfrak{R})$ and the mapping $M \to \bar{y}(M)$ in the usual manner; and this topology is called here the Zariski-topology of $\mathscr{V}(\mathfrak{R})$. After proving several lemmas, which

are mostly known in the geometric case, it is shown that, when $\mathfrak{R}$ is a regular local ring, $\bar{y}(\mathfrak{R})$ is linearly connected (this means that any two points in $\bar{y}(\mathfrak{R})$ can be joined by a sequence of rational curves in the set, finite in number); and the connectedness of $\mathscr{V}(\mathfrak{R})$ is derived from this.

Using this, and using arguments based on the idea of Hensel's lemma (applying them essentially on Chow-forms of positive cycles), the author shows that the algebraic family $\bar{Z}(\mathfrak{R})$ of positive cycles is $\overline{\mathfrak{R}}$-connected. This theorem, called the "extended principle of degeneration", is more general than his previous result contained in Ann. of Math. (2) **66** (1957), 70–79 [MR **19**, 880]. The author finally derives his general connectedness theorem, that $|\bar{Z}(\mathfrak{R})|$ is $\overline{\mathfrak{R}}$-connected, from the extended principle of degeneration without much difficulty. These results generalize Zariski's results contained in his *Theory and applications of holomorphic functions on algebraic varieties over arbitrary group fields* [Mem. Amer. Math. Soc. no. 5 (1951); MR **12**, 853]. *T. Matsusaka* (Princeton, N.J.)

1574:

Abhyankar, Shreeram. Tame coverings and fundamental groups of algebraic varieties. II. Branch curves with higher singularities. Amer. J. Math. **82** (1960), 120–178.

The notations and terminology are the same as in the review MR **21** #3428 of part I of this series of papers [same J. **81** (1959), 49–94]. The author begins with preliminary results about local rings and quadratic transforms of regular local rings (especially two-dimensional). An ideal $\mathfrak{q}$ in a regular local ring R is said to have a strong normal crossing (in R) if there exists a regular system of parameters $(x_1, \cdots, x_n)$ of R such that $\mathfrak{q} = x_1^{u(1)} \cdots x_n^{u(n)} R$; the ideal $\mathfrak{q}$ is said to have a normal crossing if it has a strong normal crossing in the completion of R; this corresponds to the geometric definition given in part I. Several results of part I are false unless the hypothesis of "normal crossing" is replaced by "strong normal crossing".

Given two principal ideals $\mathfrak{a}, \mathfrak{b}$ in a two-dimensional regular local ring R, one defines an integer $\nu(\mathfrak{a}, \mathfrak{b}; R)$ ("the strength of the singularity of $\mathfrak{a}$ on $\mathfrak{b}$ at R"), and similarly $\nu(W, W'; V)$ for two curves W, W' on a non-singular surface V. If W is an irreducible component of W', and if $\dim|W| > 1 + \nu(W, W'; V)$, there exists a quadratic transformation $f: V_1 \to V$ of V such that $f^{-1}(W')$ has only strong normal crossings. The integer $\nu(W, W'; V)$ is computed in several geometric cases.

This is used for proving the following main results on tame fundamental groups. Let V be a non-singular simply connected surface, (W_i) t irreducible curves on V, and $W = W_1 \cup \cdots \cup W_t$. (1) If, for a suitable labeling of indices, we have $\dim|W_j| > 1 + \nu(W_j, W_j \cup W_{j+1} \cup \cdots \cup W_t; V)$ for $j = 1, \cdots, t$, then $\pi^1(V - W)$ is generated by t generators, is t-step solvable, and has a weak parent group generated by t generators. (2) If $\dim|W_j| > 1 + \nu(W_j, W; V)$ for $j = 1, \cdots, t$, then $\pi^1(V - W)$ is generated by t generators, is t-step nilpotent, and has a weak parent group generated by t generators. Also several results of part I are refined in the two-dimensional case.

P. Samuel (Clermont-Ferrand)

1575:

Abhyankar, Shreeram. Tame coverings and fundamental groups of algebraic varieties. III. Some other sets of conditions for the fundamental group to be abelian. Amer. J. Math. **82** (1960), 179–190.

Let V be a simply-connected non-singular variety, and $W = W_1 \cup \cdots \cup W_t$ a pure subvariety of codimension one. We assume that, for each j, k, some non-zero integral multiple of W_j is linearly equivalent to some integral multiple of W_k, and that, either W has only strong normal crossings and $\dim|W_j| > 1$, or $\dim(V) = 2$ and $\dim|W_j| > 1 + \nu(W_j, W_j \cup \cdots \cup W_t; V)$; then $\pi^1(V - W)$ is abelian and generated by t generators. This is proved by adjoining suitable rth roots to the function field of V, so that the tame ramification is removed. Application to the tame and reduced fundamental groups of $P_n - W$ (P_n: projective space), for suitable subvarieties W.

P. Samuel (Clermont-Ferrand)

1576:

Abhyankar, Shreeram. Tame coverings and fundamental groups of algebraic varieties. IV. Product theorems. Amer. J. Math. **82** (1960), 341–364.

Let $\hat{Z}$ be the completion of the group Z of integers with respect to the topology defined by the subgroups nZ, where n is prime to the characteristic. Denote by $\overline{\pi^1}$ the inverse limit of the tame fundamental group tower. Let $W_1, \cdots, W_s, H_1, \cdots, H_t$ be distinct irreducible subvarieties of codimension 1 of a non-singular variety V; let W [resp. H] be the union of the W_i [resp. H_j]. Then, under certain conditions relating H and W, we have an exact sequence

$$0 \to \overline{\pi^1}(V - W - H) \to \overline{\pi^1}(V - W) + (\hat{Z})^t \to G \to 0,$$

where G is a finite abelian group. Application to the fundamental group of the projective plane minus a union of lines. *P. Samuel* (Clermont-Ferrand)

1577:

Abhyankar, Shreeram. Tame coverings and fundamental groups of algebraic varieties. V. Three cuspidal plane quartics. Amer. J. Math. **82** (1960), 365–373.

Let H be an irreducible quartic curve with 3 cusps in projective plane P_2 (characteristic $\neq 2, 3$). Then the tame fundamental group $\overline{\pi^1}(P_2 - H)$ is a non-abelian group of order 12, with generators a, b and relations $a^2 = 1$, $b^4 = 1$, $ba = a^2b$. This is proved by using the fact that H is the branch curve of the projection on P_2 of a cubic surface with a double line. *P. Samuel* (Clermont-Ferrand)

1578:

Abhyankar, Shreeram. Tame coverings and fundamental groups of algebraic varieties. VI. Plane curves of order at most four. Amer. J. Math. **82** (1960), 374–388.

Determination of $\overline{\pi^1}(P_2 - W)$, where W is a curve of order $\leqq 4$. If W is irreducible, this group is a cyclic group of order equal to the order of W, except if W is a quartic curve with 3 cusps [see part V]. If W is reducible, there are 10 cases, in many of which the group is infinite and unsolvable; characteristics 2, 3, 5 are sometimes excluded. The fundamental group and the reduced fundamental group of $P_2 - W$ are also determined.

P. Samuel (Clermont-Ferrand)

1579:

Herszberg, J. Types of zero-dimensional valuations of function fields of algebraic manifolds with pre-assigned valuation rings. J. London Math. Soc. **35** (1960), 177–183.

Let V be a normal variety {in the article, it is assumed to be a projectively normal variety, but we need not assume the condition, nor that V is a projective variety either}. Let P be a point on V. Then the derived normal variety V' of the quadratic dilatation of V in the usual sense is well defined. (The transformation from V onto V' is called in this article merely the quadratic transformation with P as base.) A sequence of varieties V_i $(i=1, 2, \cdots)$ is called a normal sequence if there is a point P_i on each V_i such that V_{i+1} is the derived normal variety of the quadratic dilatation of V_i with center P_i and such that P_{i+1} dominates P_i (for every $i=1, 2, \cdots$). Let Ω be the union of all the local rings of the points P_i {taken over a field K over which the points P_i are rational; in this article, it is assumed that K is the complex number field, but we need not assume it}.

The result in this article is now stated as follows. If Ω is a valuation ring, then it is of dimension zero and either it is of rank 1 (=altitude 1=Krull dimension 1) or there is a prime ideal $\mathfrak{p}$ of depth 1 such that $\Omega_{\mathfrak{p}}$ is of dimension $d-1$, where $d=\dim V_1$.

It is noted, conversely, that (1) if R is a zero-dimensional valuation ring of the function field of V_1 and if it is of rank 1, then $\Omega=R$ in the case where P_i is the center of R on V_i and (2) the same is not true in the other type of valuations if $d\geqq 3$. *M. Nagata* (Kyoto)

1580:

Severi, Francesco. Sur les irrégularités des variétés algébriques. C. R. Acad. Sci. Paris **250** (1960), 635–638.

Let A be a hypersurface on an algebraic variety V of dimension d, lying in S_{d+1} and having ordinary singularities. The author shows that the superabundance of the linear system adjoint to A is equal to the number of linearly independent $(d-1)$-ple integrals of the first kind whose integrands vanish on A. A corollary is that the adjoint system is regular if, and only if, A is topologically general; that is, if every topological cycle on V of dimension less than d is homologous to a cycle on A. {The paper contains some misprints: p. 635, line 12, read $h\leqq k$ for $h\leqq k-1$; on p. 638, in the enunciation of the main theorem, omit the abbreviation t.g.} *J. A. Todd* (Cambridge, England)

1581:

Severi, Francesco. Sur un point de la théorie des irrégularités des variétés algébriques. C. R. Acad. Sci. Paris **250** (1960), 1759.

The author proposes a revised definition of 'topologically general' subvarieties [cf. #1580]. The new definition is as follows: V_k is topologically general on V_d if the number of s-ple differential forms on V_k is equal to the number of such forms on V_d, for $s=1, 2, \cdots, k-1$. *J. A. Todd* (Cambridge, England)

1582:

Severi, Francesco. Fondamenti della geometria sulle varietà algebriche. VI. Ulteriori proprietà fondamentali delle irregolarità. Ann. Mat. Pura Appl. (4) **49** (1960), 283–298.

The author obtains a number of consequences of the following theorem, which is deduced from postulation formulae. Let V be an algebraic hypersurface with ordinary singularities, and let E be a generic hyperplane section. Then the sum of the superabundance of the system $|K+E|$ adjoint to E and the deficiency of the system cut on E by $|K+E|$ is equal to the sum of the last two irregularities of V.

The author attempts to derive from this result a number of theorems announced previously [#1580] by classical methods, the original deduction of these results resting on a theorem first proved by transcendental methods by Kodaira. To this end he introduces the following postulate: the last irregularity q_d of an irreducible non-singular V_d is equal to the number of independent $(d-1)$-cycles of V_d. But this postulate seems to be invalid, a counter-example being projective space of three dimensions. *J. A. Todd* (Cambridge, England)

1583:

Baily, Walter L., Jr. On the moduli of Jacobian varieties. Ann. of Math. (2) **71** (1960), 303–314.

Let V be a projective (algebraic) variety over the complex field, such that an open dense subset of V (in the Zariski topology) is isomorphic, as a complex analytic manifold, to a Siegel space. The points of V, with the exception of those of a proper subvariety, are thus in 1-1 correspondence $v\to A(v)$ with (isomorphism classes of) abelian varieties. The author proves the following result. There are subvarieties $R_1\subset R_0\subset R\subseteq V$ such that the conditions $v\in R-R_0$, $v\in R_0-R_1$ are equivalent, respectively, to the statements: $A(v)$ is a jacobian variety; $A(v)$ is the product of at least two jacobian varieties. Moreover, R is irreducible.

The proof relies on theta functions, period matrices, and on a criterion by T. Matsusaka for jacobian varieties [Mem. Coll. Sci. Univ. Kyoto Ser. A. Math. **32** (1959), 1–19; MR **21** #7213]. *I. Barsotti* (Providence, R.I.)

1584:

Abe, Eiichi; Kanno, Tsuneo. Some remarks on algebraic groups. Tôhoku Math. J. (2) **11** (1959), 376–384.

Let G be a connected algebraic group defined over an algebraically closed field k, and $k(G)$ the field of rational functions of G defined over k. Call a subfield F of $k(G)$ a (*)-subfield if F contains k, if $k(G)$ is separably generated over F, and if F is mapped into itself under the right translation by any rational point p over k in G. The main result of the paper is that there exists a Galois correspondence between (*)-subfields of $k(G)$ and algebraic subgroups of G defined over k. The paper also contains some remarks on conditions for a p-subalgebra of the Lie algebra $\mathfrak{g}$ of G to be algebraic. *R. Rees* (Vancouver, B.C.)

1585:

Tôgô, Shigeaki. On splittable linear groups. Rend. Circ. Mat. Palermo (2) **8** (1959), 49–76.

L'auteur dit qu'un sous-groupe G du groupe linéaire $GL(n, K)$ (K corps parfait arbitraire) est scindable si pour toute matrice $x\in G$, les parties semi-simple et unipotente de x appartiennent aussi à G. Un théorème de Borel affirme que tout groupe linéaire algébrique est scindable. L'auteur

cherche à étendre à ces groupes les propriétés établies par Borel et Chevalley pour les groupes linéaires algébriques. Ceci ne peut naturellement se faire qu'avec des hypothèses assurant que le groupe ne s'écarte pas trop de son enveloppe algébrique G^* (adhérence de G pour la topologie de Zariski). L'auteur introduit deux telles hypothèses relatives à la série dérivée et à la série centrale descendante: il dit que G est un D^∞-groupe [resp. un C^∞-groupe] si $D^\infty G^* \subset G$ [resp. $C^\infty G^* \subset G$]. Pour les groupes résolubles connexes G, il introduit aussi la condition (S): pour tout tore maximal T de G, G est produit semi-direct de T et de G_u, sous-groupe des éléments unipotents de G (propriété qui est encore un théorème de Borel pour les groupes algébriques). Ces propriétés permettent alors de généraliser aux groupes qui les possèdent la plupart des résultats connus sur les sous-groupes résolubles connexes maximaux (sous-groupes de Borel), tores maximaux et sous-groupes de Cartan des groupes linéaires algébriques. Un des théorèmes principaux de l'auteur (th. 11.4) caractérise les groupes scindables parmi les C^∞-groupes connexes comme ceux dont les sous-groupes de Borel possèdent la propriété (S). Finalement, l'auteur donne des exemples de groupes non algébriques qui rentrent dans les catégories qu'il étudie; en particulier, si K est le corps des nombres complexes, tous les groupes de Lie connexes dans $\mathrm{GL}(n, K)$ sont des C^∞-groupes.

J. Dieudonné (Paris)

LINEAR ALGEBRA

See also 1520a-b.

1586:

Souriau, Jean-Marie. ★**Calcul linéaire.** "Euclide." Introduction aux Études Scientifiques. Presses Universitaires de France, Paris, 1959. 263 pp. 22 NF.

Chapter titles: General algebras; Linear algebras; Matrix calculus; n-dimensional spaces; Multilinear algebras; Spectral properties.

Quoting from the Preface: "Le présent traité est un ouvrage de mathématiques pures, destiné à tous ceux qui doivent appliquer l'algèbre linéaire.—Le lecteur trouvera ici un exposé complet du point de vue logique, qui ne suppose pas d'autres connaissances que celles de l'enseignement secondaire.—Le présent ouvrage étudie les invariantes par le groupe linéaire. Nous envisageons la publication ultérieure d'un volume consacré aux géométries bilinéaires.—Enfin, l'étude spectrale des matrices est traitée par une méthode entièrement nouvelle, qui se décompose en deux parties: 1°, Un algorithme qui fournit notamment le déterminant, l'inverse et le polynôme caractéristique. 2°, Une décomposition en éléments simples; cette décomposition introduit les projecteurs propres et nilpotents propres de la matrice, dont la considération permet de traiter, sans recourir à la réduction de Jordan, les principales applications, théoriques et pratiques. . . ."

The author accomplishes his aims with marked economy of method and space. Together with the volume by A. Châtelet, *Arithmétique et algèbre modernes* [Presses Universitaires de France, Paris, 1954; MR **15**, 773] of the same collection "Euclide", to which frequent reference is made, the volume is practically self-contained. Each chapter contains a list of exercises in the form of problems and theorems. There is a table of contents, covering three pages, but American readers will note the absence of an alphabetic index and of a glossary of technical terms of which there are many. The almost complete absence of literature references (except to Châtelet) precludes even the slightest approach to an understanding of the historical development of the subject. It is a book in the strict Euclidean style. This can be defended, but in this country the psychological and pedagogical attitude toward mathematics follows a different trend.

A. J. Kempner (Boulder, Colo.)

1587:

Parodi, Maurice. ★**La localisation des valeurs caractéristiques des matrices et ses applications. Préface de H. Villat.** Traité de Physique Théorique et de Physique Mathématique, XII. Gauthier-Villars, Paris, 1959. xi+172 pp. 3700 fr.; $7.76.

In 1952, the author published his book *Sur quelques propriétés des valeurs caractéristiques des matrices carrées* [Gauthier-Villars, Paris; MR **14**, 236]. The aim of this book was to give a survey of certain results on characteristic roots of matrices which are based on theorems of L. Lévy, Hadamard and Minkowski, and which are treated either not at all or very superficially in the many books on matrices published during the last years. Since these results have found applications in very different fields of the sciences and engineering, it is of importance that the book is easily readable, not only for the specialist. The present volume is an enlargement to nearly three times the size of the former book. It has the same aim as the original book and is written in a similar way. There are new chapters on the classical properties of the characteristic roots, applications of the methods for the localization of the characteristic roots to criteria for stability of mathematical analogue machines, and applications to algebraic equations. In the other chapters many results obtained during the last 10 years are added. {The reference on page 31 should read vol. IX instead of vol. III and that on page 101 should read p. 309 instead of p. 2509.}

A. Brauer (Chapel Hill, N.C.)

1588:

Ostrowski, A. M. A quantitative formulation of Sylvester's law of inertia. Proc. Nat. Acad. Sci. U.S.A. **45** (1959), 740–744.

The main result of the present paper consists in the following sharpening of Sylvester's law of inertia.

Theorem 1: Let H be a hermitian matrix with characteristic roots $\lambda_1 \leqq \cdots \leqq \lambda_n$; let S be a real matrix; and let $\Lambda_1 \leqq \cdots \leqq \Lambda_n$ be the characteristic roots of $S^T H S$ (where S^T denotes the transpose of S). Then $\Lambda_k = \theta_k \lambda_k$ $(1 \leqq k \leqq n)$, where the θ_k lie between the smallest and the largest characteristic roots of $S^T S$.

It is easily shown that theorem 1 is equivalent to a result proved by the author a little earlier [Abh. Math. Sem. Univ. Hamburg **23** (1959), 60–68; MR **21** #3435; Satz 1], which enables one to compare the characteristic roots of AB with those of B; here A, B are hermitian matrices and A is non-negative. A direct proof of theorem 1, based on the Fischer-Courant minimax principle, is also given. Further extensions and generalizations are then considered; we content ourselves with quoting one of them.

Theorem 5: Let H be a hermitian and S a real matrix. Denote by $\lambda'_1, \cdots, \lambda'_n$ the characteristic roots of H; by $\Lambda'_1, \cdots, \Lambda'_n$ those of S^THS; and by $p_1, \cdots, p_n$ those of S^TS; and suppose that these roots are numbered such that $|\lambda'_1| \leqq \cdots \leqq |\lambda'_n|$, $|\Lambda'_1| \leqq \cdots \leqq |\Lambda'_n|$, $p_1 \leqq \cdots \leqq p_n$. Then, for $1 \leqq k \leqq n$,

$$p_1 \cdots p_k |\lambda'_1 \cdots \lambda'_k| \leqq |\Lambda'_1 \cdots \Lambda'_k| \leqq p_{n-k+1} \cdots p_n |\lambda'_1 \cdots \lambda'_k|.$$

As may be expected, the proof depends on the use of compound matrices. *L. Mirsky* (Sheffield)

1589:

Mulholland, H. P.; Smith, C. A. B. An inequality arising in genetical theory. Amer. Math. Monthly **66** (1959), 673–683.

In studying the existence of stable equilibria in a population subject to natural selection, the authors establish the following inequality. Let v, u be arbitrary nonzero real $k \times 1$ and $k \times k$ matrices, respectively, with all their elements non-negative, and with u symmetrical, i.e., $u^T = u$, where u^T denotes the transpose of u. Then, for any positive integer n,

$$(v^Tu^nv)(v^Tv)^{n-1} \geqq (v^Tuv)^n,$$

with equality if and only if v is a latent vector of u.
R. G. Stanton (Waterloo, Ont.)

1590:

Brenner, J. L. Relations among the minors of a matrix with dominant principal diagonal. Duke Math. J. **26** (1959), 563–567.

The author considers matrices with dominant principal diagonal. It is shown in which way the principal minor in any p rows exceeds the sum of the absolute values of the determinants of certain other minors in the same set of rows. Among important special cases are (1) the fact that in a matrix with dominant principal diagonal a principal minor exceeds in absolute value any non-principal minor on the same rows, and (2) the fact that the principal minor exceeds in absolute value the sum of the absolute values of all those non-principal minors on the same rows which involve precisely a certain fixed set of the columns of the principal minor. Even the well known fact that the whole matrix is non-singular is obtained as a special case. Another special case was given by Ostrowski [Proc. Amer. Math. Soc. **3** (1952), 26–30; MR **14**, 611]. *O. Taussky-Todd* (Pasadena, Calif.)

1591:

Scherk, Peter. On regular bilinear forms. Ann. Mat. Pura Appl. (4) **47** (1959), 391–399.

Given a regular bilinear form $\{x, y\}$ defined in a finite-dimensional vector space V over a field of characteristic $\neq 2$. One considers the mappings $x \to \varphi x = \{x, y\}$ and $x \to \psi x = \{y, x\}$. (If $\{x, y\} = x'Ay$ with a regular square matrix A, then $\varphi x = A'x$ and $\psi x = Ax$.) Let $\tau = \varphi^{-1}\psi$; then $\{\tau x, y\} = \{y, x\}$. The first part of the paper is concerned with a decomposition of V into a direct sum of subspaces, distinguished with regard to the given form in the following way: Let (x, y) and $[x, y]$ be the symmetric and skew-symmetric components of $\{x, y\}$ respectively; then $(\tau x, \tau y) = (x, y)$ and $[\tau x, \tau y] = [x, y]$. Let $V_1 = \{x \mid (x, y) = 0, \ \forall y \in V\} = \{x \mid x + \tau x = 0\}$, $V_2 = \{x \mid [x, y] = 0, \ \forall y \in V\} = \{x \mid x - \tau x = 0\}$; then $V_1 \cap V_2 = 0$. Also let $V_3 = V_1 + V_2$, $V_4 = \{y \mid y = z + \tau z, \ z \in V\}$ and define W_1, W_2, W_3 by $V_4 = (V_3 \cap V_4) + W_1$, $V_1 = (V_1 \cap V_4) + W_2$, $V = (V_1 + V_4) + W_3$. Then $V = (V_1 \cap V_4) + V_2 + W_1 + W_2 + W_3 = W_1 + V_3 + W_3$ and $[x, y]$ is found to be regular in $W_1 + W_2$. In the second part a symmetric form (x, y) and an orthogonal transformation τ are given that satisfy certain necessary conditions implied by the preceding formulas. It is then shown how all skew-symmetric forms $[x, y]$ can be constructed so that $\{x, y\} = (x, y) + [x, y]$ is a regular form such that $\{\tau x, y\} = \{y, x\}$. Finally it is indicated how the analogous problem can be solved for given τ and skew-symmetric $[x, y]$. *H. Schwerdtfeger* (Montreal)

1592:

Cohen, Eckford. The characteristic divisors of a polynomial function of a matrix. Duke Math. J. **27** (1960), 159–162.

Let $(F)_n$ denote the ring of $n \times n$ matrices over a field F and let A be a matrix of $(F)_n$ with characteristic polynomial $Q(x)$. Also let $f(x)$ be an arbitrary polynomial in the domain $F[x]$. The problem of determining the characteristic divisors of the matrix $B = f(A)$ was first solved by Kreis [Dissertation, Zürich, 1906]. In the present paper the author gives a fresh solution of the problem based on the theory of representation spaces and valid for arbitrary fields F.

One may assume without loss of generality that A is non-derogatory and that $Q(x) = P^d(x)$, where $P(x)$ is irreducible in $F[x]$. Let λ be a root of $P(x)$ and suppose that the stem field $F(\lambda)$ is of degree m over the subfield $F(\mu)$, where $\mu = f(\lambda)$. Denote by $N(z)$ the minimum polynomial of μ over F and let t be the largest positive integer $\leqq d$ such that

$$N(z) \equiv 0 \pmod{P^t(x)} \quad (z = f(x));$$

also put $d = qt + r$, $0 \leqq r < t$. The main result proved in the paper is contained in the following theorem of McCoy [Amer. J. Math. **57** (1935), 491–502]: The characteristic divisors of $B = f(A)$ are mt in number and are given by $N^{q+1}(z)$ repeated mr times and $N^q(z)$ repeated $m(t-r)$ times. *L. Carlitz* (Durham, N.C.)

1593:

Pitcher, Everett. The variation in index of a quadratic function depending on a parameter. Bull. Amer. Math. Soc. **65** (1959), 355–357.

Let $B(x, y; \lambda)$ be a real symmetric bilinear function on a real vector space X, differentiable with respect to λ on some real interval Λ for all $(x, y) \in X \times X$. The 'state' of the associated quadratic function $Q(x; \lambda) \equiv B(x, x; \lambda)$ at an admissible value of λ is defined to be the set of three numbers: (a) the 'negative index' $h(\lambda)$, defined to be the least upper bound of the dimension of planes on which $Q(x; \lambda)$ is negative definite; (b) the 'positive index' $m(\lambda)$, defined as the negative index of $-Q(x; \lambda)$; and (c) the 'nullity' $\nu(\lambda)$, defined as the dimension of the characteristic plane $N(\lambda) = \{x \mid B(x, y; \lambda) = 0 \text{ for all } y \in X\}$. For finite dimensional X, the author develops formulas for calculation of the change in $h(\mu)$ over suitable subintervals of Λ, in terms of the state numbers $h^0(\lambda)$, $m^0(\lambda)$, $\nu^0(\lambda)$ of $Q_\lambda(x; \lambda)$; e.g., if $\nu^0(\mu)$ is constant over the interval (μ_1, μ_2) except possibly at isolated values of μ which do not

include μ_1 or μ_2, then $h(\mu_2)-h(\mu_1)=\sum[h^0(\lambda)-m^0(\lambda)]$, where the sum is taken for $\mu_1<\lambda<\mu_2$. State numbers of $Q_\lambda(x;\lambda)$ can be computed respectively, as the number of negative, positive and zero roots ρ which, with $x\neq 0$, satisfy $C(\lambda)x=0$, $C(\lambda)y+C'(\lambda)x-\rho x=0$, where $C(\lambda)$ is the matrix of $Q(x;\lambda)$. The author observes that $h(\lambda)$ is the number of negative eigenvalues of $C(\lambda)$, generalizing a classical result for the case $C(\lambda)=A-\lambda I$. No proofs are given. *R. F. Rinehart* (Cleveland, Ohio)

1594:

Hsu, Chen-Jung. Some remarks on inner product in product space of unitary spaces. Proc. Japan Acad. **35** (1959), 203–208.

The two natural ways to introduce an inner product into the Grassman algebra of a unitary space give the same result. The author's other remarks are similar. {There are some minor misstatements.}

C. Davis (Providence, R.I.)

ASSOCIATIVE RINGS AND ALGEBRAS

See also 1825.

1595:

Martindale, Wallace S., III. The commutativity of a special class of rings. Canad. J. Math. **12** (1960), 263–268.

On sait qu'un anneau est commutatif si chaque commutateur $u=xy-yx$ de R satisfait la relation $u^{n(u)}=u$ pour un $n(u)\geq 2$ (Herstein). On appelle ici γ-anneau tout anneau R pour lequel $u^{n(u)}-u$ est, pour tout u, central avec un $n(u)$ convenable. On démontre alors la conjecture d'Herstein suivant laquelle tout commutateur d'un γ-anneau est central. On prouve pour cela que chaque γ-anneau de division est un corps, par recours à la théorie des corps valués et au théorème de Luroth. On voit qu'un γ-anneau semi-simple est commutatif et on donne un exemple de γ-anneau non commutatif, ensemble des matrices carrées du 3e ordre sur un corps qui sont proprement triangulaires.

J. Guérindon (Rennes)

1596:

Thierrin, Gabriel. Sur la structure d'une classe d'anneaux. Canad. Math. Bull. **3** (1960), 11–16.

A determination of all rings A for which $aA=a^2A$ for each $a\in A$. Since this property is preserved under subdirect sums, the problem is to determine the subdirectly irreducible rings in the class. These turn out to be division rings, zero rings, and rings which, additively, are the direct sum of two copies of a division ring but have an irregular multiplication.

W. G. Lister (Oyster Bay, N.Y.)

1597:

Weisfeld, Morris. On derivations in division rings. Pacific J. Math. **10** (1960), 335–343.

The main questions treated in this paper are of the Galois theory type. Thus a duality is sought between Lie rings of derivations in a ring A and subrings of A. The duality mappings are from sets of derivations to their common kernel or set of "constants", and from a subring to the set of derivations which annihilate it. A "fundamental theorem" is obtained for finite-dimensional Lie rings of restricted outer derivations in a division ring of characteristic p. As in more familiar settings, related results are obtained concerning the extension of mappings (isomorphisms and derivations), and the relation between dimensions of vector spaces determined by Lie rings of derivations and dimensions of other spaces determined by the associated rings of constants.

W. G. Lister (Oyster Bay, N.Y.)

1598:

Sakuma, Motoyoshi. On the complete tensor product of modules. J. Gakugei Tokushima Univ. **10** (1959), 3–9.

The article gives a proof of an immediate generalization of a known formula on the multiplicity in a complete tensor product of local rings. {The generalization is substantially known because it follows from the known formula, Prop. 11 in the reviewer's paper in Proc. Internat. Sympos. Algebraic Number Theory 1955, pp. 191–226, Science Council of Japan, Tokyo, 1956 [MR **18**, 637] and from the remark on the multiplicity of a module added at the end of that paper. There are some places where one can simplify the proofs. In particular, the proof of Prop. 3 can be given in a very much simpler way because E/qE and $E'/q'E'$ are vector spaces over the field K.} *M. Nagata* (Kyoto)

1599:

Northcott, D. G. A note on the coefficients of the abstract Hilbert function. J. London Math. Soc. **35** (1960), 209–214.

A local ring is called semi-regular in this article if it is a Macaulay local ring, or equivalently, if it has a distinct system of parameters. When Q is a local ring (Noetherian) of altitude d (=Krull dimension d, =rank d) and $\mathfrak{q}$ is a primary ideal belonging to the maximal ideal, then the length of $Q/\mathfrak{q}^{n+1}$ is a polynomial in n for large values of n, which is expressed as

$$e_0(\mathfrak{q})\binom{n+d}{d}-e_1(\mathfrak{q})\binom{n+d-1}{d-1}+\cdots$$
$$+(-1)^i e_i(\mathfrak{q})\binom{n+d-i}{d-i}+\cdots+(-1)^d e_d(\mathfrak{q}).$$

The $e_i(\mathfrak{q})$ are called normalized Hilbert coefficients.

The main result in this article is the following. If Q is semi-regular (=Macaulay), then

$$e_1(\mathfrak{q})\geq e_0(\mathfrak{q})-\text{length}\,(Q/\mathfrak{q})\ (\geq 0).$$

As an immediate consequence of this result, it is remarked that $\mathfrak{q}$ is generated by a system of parameters (under the assumption that Q is Macaulay) if and only if $e_1(\mathfrak{q})=0$ (and in that case $e_i(\mathfrak{q})=0$ for all $i=1, 2, \cdots, d$).

M. Nagata (Kyoto)

1600:

Northcott, D. G. A generalization of a theorem on the content of polynomials. Proc. Cambridge Philos. Soc. **55** (1959), 282–288.

Si P est un polynôme à une variable x, à coefficients en un anneau R, commutatif, avec élément unité, on désigne par $C(P)$ l'idéal engendré en R par les coefficients de P. On sait que pour tout couple de polynômes f et g, on a, en posant $h=fg$, $C(f)C^{d+1}(g)=C(h)C^d(g)$ pour l'entier d

assez grand. On rappelle ici la démonstration de ce fait, due en substance à Artin et reposant sur l'utilisation de la formule d'interpolation de Lagrange. Une généralisation est alors obtenue au moyen de la notion de (M, G)-ensemble: M étant un module unitaire sur l'anneau R et G étant un groupe additif, un (M, G)-ensemble est une famille $\{u_\sigma\}_{\sigma\in G}$ d'éléments de M, nuls sauf un nombre fini d'entre eux. Le "contenu" (content) de (M, G) est par définition le sous-groupe engendré par les u_σ dans le groupe additif sous-jacent à M (donc considéré comme module sur les entiers rationnels). De plus si $\{u_\sigma\}$ est un (M, G)-ensemble et $\{a_\sigma\}$ un (R, G)-ensemble, leur G-produit sera par définition le (M, G)-ensemble $\{v_\sigma\}$ avec $v_\sigma = \sum_{\rho\tau=\sigma} u_\rho a_\tau$ $(\sigma \in G)$. Enfin on désigne pour tout sous-groupe additif N de M et tout sous-groupe additif B de R par NB le sous-groupe additif de M engendré par les produits wb $(w \in N,\ b \in B)$. Le résultat obtenu est le suivant: si un (M, G)-ensemble de contenu U et un (R, G)-ensemble de contenu V ont un G-produit de contenu A, alors pour l'entier k assez grand, on a $UA^{k+1} = VA^k$, dès que le groupe donné G est sans torsion. On se ramène au cas des polynômes et on montre que l'existence de l'élément unité de R est superflue; le résultat subsiste enfin si G est plus généralement un sous-demi-groupe d'un groupe sans torsion. *J. Guérindon* (Rennes)

1601:

Northcott, D. G. The reduction number of a one-dimensional local ring. Mathematika **6** (1959), 87–90.

Dans cette analyse et la suivante, A désigne un anneau local de dimension 1, et $\mathfrak{m}$ son idéal maximal; on suppose que $\mathfrak{m}$ ne se compose pas uniquement de diviseurs de zéro. Pour n grand, la longueur $l(A/\mathfrak{m}^n)$ est un polynôme $en - r$; on appelle r le nombre de réduction de A. Si R est le premier anneau de voisinage de A [Northcott, Proc. Cambridge Philos. Soc. **53** (1957), 43–56; MR **18**, 462], on a $l_A(R/A) = r$; ainsi $r = 0$ si et seulement si A est régulier. Calcul de r lorsque A est une intersection complète. Étude des nombres de réduction des anneaux locaux des voisinages supérieurs de A.

P. Samuel (Clermont-Ferrand)

1602:

Kirby, D. The defect of a one-dimensional local ring. Mathematika **6** (1959), 91–97.

Soit $\mathfrak{q}$ un idéal primaire pour $\mathfrak{m}$; pour n grand, on a $l(A/\mathfrak{q}^n) = e(\mathfrak{q})n - r(\mathfrak{q})$; on appelle $r(\mathfrak{q})$ le nombre de réduction de $\mathfrak{q}$. Si A (ou son complété) admet des éléments nilpotents, les entiers $r(\mathfrak{q})$ ne sont pas majorés. On a $r(\mathfrak{q}) \geqq 0$, et $r(\mathfrak{q}) = 0$ si et seulement si une puissance de $\mathfrak{q}$ est un idéal principal. Si la clôture intégrale A' de A est un A-module de type fini, on a $0 \leq r(\mathfrak{q}) \leq l_A(A'/A)$; de plus, si $\mathfrak{q}$ est aussi un idéal de A', on a $r(\mathfrak{q}) = l_A(A'/A)$ (N.B.: c'est ce dernier entier qu'on appelle le défaut de A).

P. Samuel (Clermont-Ferrand)

1603:

Jans, J. P. Some remarks on symmetric and Frobenius algebras. Nagoya Math. J. **16** (1960), 65–71.

The concepts of Frobenius and symmetric algebras of infinite dimension over a field have been defined by the author [Ann. of Math. (2) **69** (1957), 392–407; MR **21** #3464]. It is shown first that the algebra of all transformations in an infinite-dimensional vector space is not Frobenius while that of transformations of finite rank is symmetric. After remarks on tensor products of Frobenius (symmetric) algebras and normalizations of inner products, remarks on polynomial and related algebras are given, including that a polynomial algebra can be made a symmetric algebra in an uncountable number of ways. *T. Nakayama* (Nagoya)

1604:

Iizuka, Kenzo. On the Jacobson radical of a semiring. Tôhoku Math. J. (2) **11** (1959), 409–421.

After introducing the concepts of right A-semimodule and representation semimodule for a semiring A, the author defines the radical of A as the intersection of the right-annihilator ideals (in A) for all the irreducible representation semimodules of A. He then proves that this radical coincides with the ones introduced by S. Bourne [Proc. Nat. Acad. Sci. U.S.A. **37** (1951), 163–170; MR **13**, 7] and by S. Bourne and H. Zassenhaus [ibid., **44** (1958), 907–914; MR **20** #3899]. A few basic properties of the radical are also derived, and some examples are exhibited. (Here a semiring is commutative relative to addition and has a zero element.) *W. E. Deskins* (E. Lansing, Mich.)

1605:

Rocos, Pant. Algèbre-anneaux. Prakt. Akad. Athēnōn **32** (1957), 308–318. (Greek. French summary)

An algebra-ring is defined to be a set Δ with two sets of binary operations $\phi_1, \phi_2, \cdots, \phi_m$ and $f_1, f_2, \cdots, f_m$. For each ϕ_i, Δ is a commutative semigroup and there is a common unity element e such that $\phi_i(a, e) = a$ for every ϕ_i and every a in Δ. The set Δ is also a semigroup under every f_j and every f_j is distributive with every ϕ_i. A left ideal J of Δ is defined by the properties: (1) if α, β are in J, $\phi_i(\alpha, \beta)$ is in J for each i; and (2) if α is in J and δ in Δ, $f_j(\alpha, \delta)$ is in J for each j. A large number of examples of algebra-rings and of ideals are given. Some results analogous to those known for rings are obtained for residue classes relative to a regular equivalence relation, ideals, homomorphisms, and isomorphisms.

L. A. Kokoris (Chicago, Ill.)

NON-ASSOCIATIVE RINGS AND ALGEBRAS

1606:

Kleinfeld, Erwin. Rings of (γ, δ) type. Portugal. Math. **18** (1959), 107–110.

Algebras of (γ, δ) type were defined by Albert [see Portugal. Math. **8** (1949), 23–36; MR **11**, 316]. In this paper it is proved that a ring of type (1, 0), (1, 1), (−1, 0), or (−1, 1) is associative if and only if $(x, y, x)^2 = 0$ implies $(x, y, x) = 0$ for all elements x, y of the ring. It follows that if the ring is without nilpotent elements it must be associative. *L. A. Kokoris* (Chicago, Ill.)

1607:

Springer, T. A. On a class of Jordan algebras. Nederl. Akad. Wetensch. Proc. Ser. A **62** = Indag. Math. **21** (1959), 254–264.

The author gives a characterization, analogous to the

characterization of the octave algebras by quadratic forms permitting composition, of the exceptional Jordan algebras (in characteristic $\neq 2, 3$). These latter may be defined as central simple Jordan algebras which, on extending the base field, become algebras of 3×3 hermitian matrices over an octave algebra. Denoting by $t(x)$ the trace of such a matrix, $Q(x)$ the quadratic form $\frac{1}{2}t(x^2)$ with associated bilinear form (x, y), and e the unit element of the Jordan algebra, we have (1) $Q(x^2)=Q^2(x)$ if $(x, e)=0$, (2) $(xy, z)=(x, yz)$, (3) $Q(e)=3/2$ (and $Q(x)$ is non-degenerate). Conversely, the author proves: a simple, commutative algebra with a (non-degenerate) quadratic form Q satisfying (1), (2), (3) and possessing zero divisors is isomorphic with a Jordan algebra of 3×3 hermitian matrices over a composition algebra.

(A composition algebra is an octave algebra, quaternion algebra, two-dimensional algebra, or one-dimensional algebra; i.e., defined by a quadratic form permitting composition.)

On replacing (3) by $Q(e)=1$, one characterizes instead the Jordan algebra of a quadratic form, i.e., the multiplication is $xy=(x, y)e$, where $\frac{1}{2}(x, x)$ is the quadratic form. Some other results are also obtained.

B. Harris (Evanston, Ill.)

HOMOLOGICAL ALGEBRA

See also 1566.

1608:

Bokštein, M. F. A theorem on universal coefficients for homology groups of chain complexes without torsion. Izv. Akad. Nauk SSSR. Ser. Mat. **23** (1959), 529–564. (Russian)

This paper contains detailed proofs of results announced in the author's notes, C. R. Acad. Sci. Paris **247** (1958), 259–261, 396–398 [MR **21** #2680a, b].

P. J. Hilton (Birmingham)

1609:

Golod, E. S. On a question of homological algebra. Uspehi Mat. Nauk **14** (1959), no. 4 (88), 157. (Russian)

The author gives an example of a finitely generated projective Z_p-module which is not free. This answers the question stated by H. Cartan and S. Eilenberg [*Homological algebra*, Princeton Univ. Press, 1956; MR **17**, 1040; cf. p. 241; see also D. S. Rim, Ann. of Math. (2) **69** (1959), 700–712; MR **21** #3474].

E. Lluis (Mexico City)

1610:

Takasu, Satoru. Relative homology and relative cohomology theory of groups. J. Fac. Sci. Univ. Tokyo. Sect. I **8** (1959), 75–110.

This theory is apparently not related to the relative homology of modules developed by Hochschild and Adamson. For a group Π and a subgroup π, define $I=\mathrm{Ker}(Z(\Pi)\otimes_{Z(\pi)}Z\to Z)$ and set $H_q(\Pi, \pi: A)=\mathrm{Tor}_{q-1}^{Z(\Pi)}(I, A)$, $H^q(\Pi, \pi: A)=\mathrm{Ext}_{Z(\Pi)}^{q-1}(I, A)$ for $q\geq 1$. If $\pi=\{1\}$, these reduce to the ordinary homology and cohomology groups $H_q(\Pi: A)$ and $H^q(\Pi: A)$. There is an exact homology sequence

$$\cdots\to H_q(\pi: A)\to H_q(\Pi: A)\to H_q(\Pi, \pi: A)\to H_{q-1}(\pi: A)\to\cdots$$

and a corresponding exact cohomology sequence. These groups can be computed using Eilenberg-Maclane cochains; e.g., if C^q is the group of q-cochains on Π which vanish on π and $\delta^q: C^q\to C^{q+1}$ is the ordinary coboundary operator, then $H^q(\Pi, \pi: A)=\mathrm{Ker}\,\delta^q/\mathrm{Im}\,\delta^{q-1}$. Functors $I(A)$ and $J(A)$ are defined (much as I is defined above; $I=I(Z)$) such that $H_q(A)=H_{q-1}(I(A))$ and $H^q(A)=H^{q-1}(J(A))$. In case Π is finite, this imbeds the homology sequence in a complete homology sequence $\{\hat{H}_q;\ -\infty<q<\infty\}$. There is a similar complete sequence for cohomology, which apparently does not coincide with the homology sequence as in the absolute case $\pi=\{1\}$.

Tate's theorem generalizes thus: If $\hat{H}^1(\pi: A)=0$ and $\hat{H}^2(\pi: A)=Z/[\pi: 1]Z$ for every subgroup π of a finite group Π, then $\hat{H}_q(\Pi, \pi: A)=\hat{H}_{q+2}(\Pi, \pi: Z)$ and $\hat{H}^q(\Pi, \pi: A)=H^{q-2}(\Pi, \pi: Z)$ for all q and all subgroups π of Π.

The author characterizes $H^3(\Pi, \pi: A)$ as the group of (Π, π)-kernels with center A, modulo extensible ones; the definitions are slightly too long to give here.

Let $F\subset K\subset L$ be fields with L Galois over F and define a Brauer group as the group of classes of central simple K-algebras A such that all automorphisms of L over F are extensible to $A\otimes_K L$. This group modulo the subgroup of classes of scalar extensions of F-algebras is isomorphic to a subgroup of $H^2(\Pi, \pi: L^*)$ (here Π and π denote the Galois groups of L over F and over K respectively). This subgroup is the union of the kernels of the lift mappings determined by $L\to N$, with N ranging over the extensions of L which are Galois over F. The author also generalizes the Hochschild-Serre spectral sequence to a relative one:

$$H^p(\Pi/\rho, \pi/\rho: H^q(\rho: A))\underset{p}{\Rightarrow} H^n(\Pi, \pi: A),$$

where ρ is a subgroup of π and is normal in Π.

D. Zelinsky (Berkeley, Calif.)

GROUPS AND GENERALIZATIONS

See also 1585, 1899, 1900.

1611:

Rabin, Michael O. Recursive unsolvability of group theoretic problems. Ann. of Math. (2) **67** (1958), 172–194.

This paper is a sizable contribution to group theory. A wide class of problems concerned with the recognition of certain properties of finitely presented groups are shown to be recursively unsolvable; in particular the well-known isomorphism problem is shown to be unsolvable. This work is quite independent of the similar work of Adyan [Dokl. Akad. Nauk SSSR **103** (1955), 533–535; **117** (1957), 9–12; Trudy Moskov. Mat. Obšč. **6** (1957), 231–298; MR **18**, 455; **20** #2370, #2371]. Indeed, the author achieved his results without knowledge of the existence of Adyan's 1955 announcement. The argument is extremely elegant; the style is leisurely with much comment directed at the non-logician so as to make quite clear exactly what is meant by saying that a problem is recursively unsolvable.

It is shown that if this certain class of problems about groups were solvable, then a known unsolvable problem, the word problem for a certain finitely presented group (see end of review for references), would be solvable. The plan of the author's argument is that originated by Markov in his demonstration of the corresponding result

for semi-groups without cancellation [Dokl. Akad. Nauk SSSR **77** (1951), 19–20, 953–956; MR **12**, 661; **13**, 4 (Markov's argument can be understood completely from a review by Andrzej Mostowski in J. Symb. Logic **17** (1952), 151); Trudy Mat. Inst. Steklov. no. 42 (1954); MR **17**, 1038]. Markov's argument was previously adapted to show the corresponding result for semi-groups with cancellation, simultaneously and independently, by John Addison and Walter J. Feeney in their doctoral dissertations [Univ. of Wisconsin, 1952; Catholic Univ. of America, 1952].

The present article can be pleasantly read by a group-theorist not acquainted with the literature of decision problems. To be convinced of the main theorem one must (a) understand, at the intuitive level, the notion of an effective process; and (b) accept the fact that there has been exhibited a finite presentation of a group with unsolvable word problem. But aside from (a) and (b), no non-group-theoretic demands are made of the reader. To follow the author's reduction argument itself the reader need not be familiar with the precise technical definition of effective process nor with a finite presentation of a group having unsolvable word problem. But perhaps the most welcome feature of the reduction argument for the group-theorist is its being framed in terms of free products of groups with amalgamations, so that the reader does not have to master a battery of lemmas about word cancellations.

We shall state the main result exactly and outline its demonstration. It is an easy matter to specify precisely the groups involved without recourse to sophisticated group theory. The notion of a decision problem and of the recursive solvability or unsolvability of such a problem is taken for granted. The notion of an FPG (finite presentation of a group) made up of a finite number of generators and a finite number of defining relations is also assumed, as well as such closely related notions of a word on the generators, of a relation holding in the presentation, of the group presented by the presentation and of the word problem for the presentation. In part for brevity we make a few (hopefully, non-confusing) changes in the author's terminology and account. Let G_Π be the group presented by the FPG Π. The group G is f.p. (finitely presentable) if there is an FPG Π such that G is G_Π. If P is a property of f.p. groups preserved under isomorphism, then P is a Markov property of groups if: there is an FPG, Π_1, such that G_{Π_1} has property P; there is an FPG, Π_2, such that G_{Π_2} cannot be embedded in any f.p. group having property P. The author's main result, i.e., Theorem 1.1, p. 176, may then be stated as follows: For any Markov property of groups, P, it is recursively unsolvable to determine of an arbitrary FPG, $\tilde{\Pi}$, whether or not $G_{\tilde{\Pi}}$ has property P. For the outline proof, let P be any Markov property; let Π_0 be any FPG with unsolvable word problem; let Π_1 and Π_2 be as in the definition of Markov property of groups just given. Let Π be the FPG whose generators are the generators of Π_0 and of Π_2, and whose defining relations are the defining relations of Π_0 and Π_2. (Thus G_Π is the free product of G_{Π_0} and G_{Π_2}.) Clearly (1) G_{Π_2} is embedded in G_Π. Moreover, G_{Π_0} is embedded in G_Π and (2) the word problem for Π is unsolvable. Where w is an arbitrary word on the generators of Π—we now take these generators to be x_1, x_2, $\cdots$, x_n—let Π_w be the FPG obtained by adjoining to Π x_{n+1}, t, a, s, b, c, and d as additional generators as well as the following, using u as an abbreviation for $x_{n+1}wx_{n+1}^{-1}w$, as additional defining relations: $ut=t^2u$; $ta=a^2t$; $us=s^2u$; $sb=b^2s$; $a=c$; $x_ib^iab^{-i}=d^icd^{-i}$ $(i=1,\cdots,n+1)$; $b^{n+2}aba^{-1}b^{-n-2}=d^{n+2}cdc^{-1}d^{-n-2}$. (The presence of both a and c together with the defining relation $a=c$ is an expositional trick.)

The author now demonstrates [cf. Markov, loc. cit. or Mostowski, loc. cit.] that (3) if $w=1$ in Π, then G_{Π_w} is the trivial group; (4) if $w\neq 1$ in Π, then G_Π is embedded in G_{Π_w}. The demonstration of (3) is completely trivial. The demonstration of (4) is the crux of the entire paper and summarization of that argument will not be attempted in this review. Suffice it to say that it is here where the theory of free products of groups with amalgamations is brought to bear with such finesse. Let $\Pi(w)$ be the FPG whose generators are the generators of Π_1 and of Π_w, and whose defining relations are the defining relations of Π_1 and Π_w. (Thus $G_{\Pi(w)}$ is the free product of G_{Π_1} and G_{Π_w}.) Clearly (5) G_{Π_w} is embedded in $G_{\Pi(w)}$ and (6) if G_{Π_w} is the trivial group, then $G_{\Pi(w)}$ and G_{Π_1} are isomorphic. Now we can assert the reduction lemma (7): $w=1$ in Π if and only if $G_{\Pi(w)}$ has property P. For if $w=1$ in Π, then by (3), (6), and the definition of Π_1, $G_{\Pi(w)}$ has property P; if $w\neq 1$ in Π, then by (1), (4) and (5), G_{Π_2} is embedded in $G_{\Pi(w)}$—and hence $G_{\Pi(w)}$ does not have property P by the definition of Π_2. The main result is now immediate by (2) and (7).

If in the above construction Π_0, Π_1 and Π_2 each has a small number of generators and defining relations, so also has each presentation $\Pi(w)$. Thus for many important special cases of P, e.g., being abelian, trivial, finite, each $\Pi(w)$ has about ten generators and forty-five non-trivial defining relations if Π_0 is taken to be a known FPG with an unsolvable word problem and having two generators and thirty-two non-trivial defining relations [Boone, Ann. of Math. (2) **70** (1959), 207–265]. If one applies Higman, B. H. Neumann and H. Neumann's two-generator embedding result [J. London Math. Soc. **24** (1949), 247–254; MR **11**, 322], to the author's argument, i.e., so embed G_Π and use the finite presentation of this two-generator extension of G_Π in place of Π in continuing the construction, this kind of result is usually sharpened in that the number of defining relations is decreased. (Note that (1) and (2) above are valid with this new version of Π.) Alternatively, one can choose Π_0 so that, using the letter "w" as parameter, the generic presentation $\Pi(w)$ can easily be written out explicitly in a few minutes' time. Various compromises between the aim of having a small number of defining relations versus that of simplicity of defining relations are possible in the construction of $\Pi(w)$ for these important special cases of P.

Regarding the author's discussion at the top of p. 173, note that his main result does not imply the corresponding results of Markov, Addison, and Feeney in general but only that special case of their theorems in which the Markov property, P, of semi-groups (without cancellation, for Markov's result; with cancellation, for Addison and Feeney's result) is a Markov property of groups; this is not the case, for example, if P is the property of being a group or being embeddable in a group. On the other hand, whether or not the author's argument can be amended in some simple fashion so as to obtain these earlier results is an open question.

Certain natural problems about groups, e.g., "Is a given

FPG simple?", are shown to be unsolvable as easy consequences of the main result, without a demonstration that the property considered is a Markov property of groups. It is also shown that (Theorem 3.2) "every infinite system of computable isomorphism invariants is not complete" and that (Theorem 3.3) the set of all FPG's presenting the same group as a given FPG is recursively enumerable. While Theorem 3.3 itself would seem well-known, the author's recursive enumeration in terms of Tietze transformations is extremely neat and should help to clarify the notion of recursive enumerability for the non-logician. The decision problems which have arisen naturally in mathematics usually have the property that either the questions with affirmative answers or those with negative answers are easily seen to be recursively enumerable. But the following query raised by J. H. C. Whitehead is of interest: Are the FPG's with solvable word problem recursively enumerable?

Finally, the reviewer would like to note that he has carefully verified the author's argument for his and Adyan's important result. There are no slips even in the most minute details. In l. −9 (not counting the footnote), p. 180, for x_n read x_{n+1}. In l. −13, p. 181, for b^{-1} read b^{-4}. In l. 7, p. 182, for d_i read d^i. Note that Π_0 is given a new definition on p. 183, distinct from that on p. 177.

The following would seem to be a complete bibliography of articles specifying finitely presented groups for which the word problem is unsolvable. P. S. Novikov, Trudy Mat. Inst. Steklov. no. 44, 1955=Amer. Math. Soc. Transl. (2) **9**, 1–122 [MR **17**, 706; **19**, 1158]; the argument uses A. M. Turing, Ann. of Math. (2) **52** (1950), 491–505 [MR **12**, 239]; corrections to Turing's article appear in W. W. Boone, same Ann. **67** (1958), 195–202 [MR **19**, 1158]. Boone, Nederl. Akad. Wetensch. Proc. Ser. A **57** (1954), 231–237, 492–497; **58** (1955), 252–256, 571–577; **60** (1957), 22–27, 227–232 (the last two parts of this series revise the earlier parts so as to give the desired result) [MR **16**, 564; **20** #5230, #5231]. J. L. Britton, Proc. London Math. Soc. (3) **8** (1958), 493–506, taken together with Britton, Proc. Glasgow Math. Assoc. **3** (1957), 68–90. Novikov and S. I. Adyan, Z. Math. Logik Grundlagen Math. **4** (1958), 66–88 [MR **20** #7052] (this article alters Novikov's earlier argument to remove the dependence on Turing, loc. cit.). W. W. Boone, Ann. of Math. (2) **70** (1959), 207–265. Graham Higman, submitted to Philos. Trans. Roy. Soc. London. Ser. A.

W. W. Boone (Urbana, Ill.)

1612:

Roman T. Symmetries of four-dimensional border ornaments. Dokl. Akad. Nauk SSSR **128** (1959), 1122–1124. (Russian)

The author lists 179 groups of symmetries of four-dimensional border ornaments. These are all elementary Abelian 2-groups. There are 8 of order 16, 64 of order 8, 84 of order 4, 12 of order 2, and the identity group.

Marshall Hall, Jr. (Pasadena, Calif.)

1613:

Crouch, R. B. Characteristic subgroups of monomial groups. Pacific J. Math. **10** (1960), 85–89.

A determination of the characteristic subgroups of those monomial groups for which the normal subgroups have already been determined [cf. R. B. Crouch and W. R. Scott, Proc. Amer. Math. Soc. **8** (1957), 931–936; MR **19**, 728].

Graham Higman (Chicago, Ill.)

1614:

Boyer, D. L.; Walker, E. A. Almost locally pure Abelian groups. Pacific J. Math. **9** (1959), 409–413.

An abelian group G is called almost locally pure (a.l.p.) if for every finite set of elements of G there exists a finitely generated pure subgroup of G which contains this finite set as a subset. The paper gives a preliminary investigation of a.l.p. groups. Some of the results: for primary groups a.l.p. means no elements of infinite height; subgroups of a.l.p. groups are a.l.p.; a countable a.l.p. group is a direct sum of cyclic groups; if H is pure in G and if H and G/H are a.l.p., then G is a.l.p.; every group has a maximal pure a.l.p. subgroup.

J. de Groot (Amsterdam)

1615:

Kemperman, J. H. B. On small sumsets in an abelian group. Acta Math. **103** (1960), 63–88.

Let A, B be finite subsets of an additive group G and let $[C]$ denote the number of elements in a subset C of G. If G is cyclic of prime order it is known that $[A+B] \geq [A]+[B]$ except in the following cases: (i) $A+B=G$; (ii) A is the complement of a translate of $-B$; (iii) A, B are in arithmetic progression with the same common difference.

In an arbitrary abelian G, let (A, B) be called a "critical pair" if $[A+B]<[A]+[B]$. By a complicated analysis based on the methods of Kneser and van der Corput, it is shown that if (A, B) is a critical pair not of types (i), (ii), (iii) there is a non-unit subgroup H such that $A+B+H=A+B$ and (A, B) remains critical after applying the natural homomorphism $G \to G/H$. This leads to a description of all critical pairs, essentially as those which can be built up from the "elementary pairs" satisfying (i), (ii), (iii) in subgroups and factor groups of G. The complete description (Theorem 5.1) would be rather complicated to reproduce here.

In the final section the author derives some results about the number of representations of an element $c \in A+B$ in the form $a+b$.

A. M. Macbeath (Dundee)

1616:

Kolettis, George, Jr. Direct sums of countable groups. Duke Math. J. **27** (1960), 111–125.

The author generalizes the description of countable reduced primary abelian groups in terms of Ulm invariants [see, e.g., Kaplansky, *Infinite abelian groups*, Univ. of Michigan Press, Ann Arbor, 1954; MR **16**, 444] to the case of reduced primary abelian groups which are direct sums of countable groups (arbitrarily many summands being allowed). That is, let G be a direct sum of countable abelian groups, all of whose elements have finite orders which are powers of a fixed prime p, such that for no subgroup $S \neq 0$ of G is $pS=S$; let P be the subgroup of all $g \in G$ with $pg=0$; for each ordinal number α define G_α by $G_0=G$; $G_{\alpha+1}=pG_\alpha$, and $G_\alpha=\bigcap_{\beta<\alpha} G_\beta$ if α is a limit ordinal. Then $P_\alpha=P \cap G_\alpha$ is a vector space over the field of p elements, and the cardinal numbers

$$f_G(\alpha) = \dim(P_\alpha/P_{\alpha+1})$$

are well defined. These are the Ulm invariants of G and characterize G to within isomorphism. Since $G_\alpha=0$ for all uncountable ordinals α, f_G is determined by its values on the countable ordinals. The author gives a characterization of those cardinal-valued functions on the countable

ordinals which can arise as Ulm functions of the class of groups under consideration. The corresponding results for countable groups are used in the proofs.

G. B. Seligman (New Haven, Conn.)

1617:

Pic, Gheorghe. Sopra un teorema di P. Hall. Atti Accad. Naz. Lincei. Rend Cl. Sci. Fis. Mat. Nat. (8) **26** (1959), 451–457.

Let P be a property of groups which is inherited by subgroups, by homomorphic images, and by direct products of arbitrarily many groups that have it (this is not the author's formulation). If there is a minimal normal subgroup modulo which a given group has P, then it is fully invariant; and descending chains are considered that correspond to the lower central series. The maximal normal subgroup with P gives rise to a dual series, corresponding to the upper central series. With these series a "non-inclusion" theorem of P. Hall [Proc. London Math. Soc. (2) **36** (1933), 29–95; Theorem 2.49, p. 51] is generalized, and the generalized theorem is dualized.

{The author gives (p. 452, penultimate paragraph) three examples purporting to be of properties P of the kind he considers. The first example does not fit his formulation because it is not a property of groups, but a relative property of elements or subgroups of a group. The second example is invalidated by the objections raised by K. A. Hirsch in MR **17**, 11, against an earlier paper by the same author [Acad. R. P. Romîne. Bul. Şti. Secţ. Şti. Mat. Fiz. **9** (1954), 199–221].}

B. H. Neumann (Manchester)

1618:

Hall, P. On the finiteness of certain soluble groups. Proc. London Math. Soc. (3) **9** (1959), 595–622.

The main result in this paper is the following: (1) Every finitely generated abelian-by-nilpotent group is residually finite. (A group is abelian-by-nilpotent if it possesses an abelian normal subgroup with nilpotent quotient group; and a group is residually finite if it contains a family of normal subgroups, each of finite index, and where the intersection of all groups in the family is the trivial subgroup.) In particular, (1) implies that every finitely generated metabelian group is residually finite and hence a minimal normal subgroup of such a group must be finite. The situation is very different in finitely generated soluble groups of derived length 3: (2) There exists a group of this latter type containing a minimal normal subgroup isomorphic with the direct sum of $\aleph_0$ copies of the additive group of rational numbers.

Outline of the proof of (1). Step 1: Call a group 'monolithic' if the intersection of all its (non-trivial) normal subgroups is itself non-trivial. Then (1) is trivially equivalent with: (1*) Every monolithic finitely generated abelian-by-nilpotent group is finite. Step 2: Let G be monolithic finitely generated abelian-by-nilpotent and B be an abelian normal subgroup with $H=G/B$ nilpotent. If $B=1$, then (1*) is easily proved. If $B\neq 1$, then B contains M, the intersection of all normal subgroups $\neq 1$ of G. Now M is isomorphic to the additive group of a vector space over either (i) GF[p], the field of p elements, or (ii) $\mathfrak{r}$, the field of rational numbers. The core of the proof of (1*) consists in showing that (ii) cannot occur (step 3 below) and that, in case (i), the vector space in question has finite dimension (step 4 below). Hence M is finite and now the finiteness of G follows by a direct, one page long, argument. Step 3: Given any abelian group A, let A_0 be the (free) subgroup generated by a maximal set of linearly independent elements, so that A/A_0 is a torsion group. If π is a given set of prime numbers, let $\mathfrak{A}(\pi)$ be the class of all abelian groups A in which A_0 can be chosen such that A/A_0 is a π-torsion group. Moreover, let $\mathfrak{B}$ be the class of all finitely generated modules over polycyclic groups. (A group is polycyclic if it can be obtained from the unit subgroup by a finite number of cyclic extensions.) (3) Every abelian group B in $\mathfrak{B}$ belongs to $\mathfrak{A}(\pi)$ for some finite set π. Then every subgroup of B is still in $\mathfrak{A}(\pi)$ and hence B cannot contain a copy of $\mathfrak{r}$ (since $\mathfrak{r}$ is in $\mathfrak{A}(\pi_1)$ where π_1 is the set of all primes). The group B of step 2 is in $\mathfrak{B}$ (because H is polycyclic) and thus case (ii) is ruled out. Step 4: Abelian groups are the same as $\mathfrak{j}$-modules, where $\mathfrak{j}$ is the ring of rational integers. If J is now any principal ideal domain and π is any given set of primes in J, one can define the class of J-modules $\mathfrak{A}(J,\pi)$ by the procedure given in step 3 but using J instead of $\mathfrak{j}$. Thus $\mathfrak{A}(\pi)=\mathfrak{A}(\mathfrak{j},\pi)$. If J is the group algebra over the arbitrary field $\mathfrak{k}$ of the infinite cyclic group on t, if H is a polycyclic group and z is an element in the centre of H, then $t\to z$ yields a $\mathfrak{k}$-algebra homomorphism of J into $\mathfrak{k}H$ by means of which any $\mathfrak{k}H$-module can be made into a J-module. (4) If M is a finitely generated $\mathfrak{k}H$-module, then M, as J-module, belongs to $\mathfrak{A}(J,\pi)$ for some finite set π. It follows (as in step 3) that M cannot contain a submodule isomorphic with the quotient field of J. This is the crucial fact needed for the proof of (5): if $\mathfrak{k}$ is an absolutely algebraic field of characteristic p and V is an irreducible representation space over $\mathfrak{k}$ for the finitely generated nilpotent group H, then V is finite-dimensional. This theorem can be applied in the situation of step 2, with $\mathfrak{k}=\mathrm{GF}[p]$ and $V=M$, and it shows that M must be finite.

The author conjectures that (5) remains true when H is an arbitrary polycyclic group. If this were so, then (1) could be extended to finitely generated abelian-by-polycyclic groups. (That every polycyclic group is itself residually finite is due to K. A. Hirsch [Proc. London Math. Soc. (2) **49** (1946), 184–194; MR **8**, 132].) But the condition on $\mathfrak{k}$ in (5) is indispensable: (6) If G is a polycyclic group having no abelian subgroup of finite index and $\mathfrak{k}$ is a field of characteristic p but not absolutely algebraic, then G has an infinite-dimensional irreducible representation space over $\mathfrak{k}$. On the other hand, (7) if G is finitely generated abelian-by-finite, then the irreducible representations of G over any field are all of finite degree.

As applications, the author proves two theorems. (8) If G has a normal ZA-subgroup K such that G/K is finitely generated nilpotent, then all the chief factors of G are finite abelian; and every maximal subgroup of G is of finite index in G. (A ZA-group is one in which the upper central series terminates at the group itself.) (9) If G is finitely generated nilpotent-by-abelian-by-finite, then the Frattini subgroup of G is nilpotent. {This last theorem has recently been considerably improved by the author. His new results will be published soon.}

K. Gruenberg (London)

1619:

Wielandt, Helmut. Sylowtürme in subnormalen Untergruppen. Math. Z. **73** (1960), 386–392.

Let π be a set of primes, π^* the same set in a fixed linear

order. A group, as usual, is called a π-group if its order is divisible only by primes in π; a π-group is called a π^*-group if it has a Sylow tower of the type indicated by the order of π^*. That is, if $p_1, \cdots, p_r$ are the primes dividing its order, in the order in which they occur in π^*, it has a Sylow p_i-subgroup P_i, for $i=1, 2, \cdots, r$, such that, for $i=1, 2, \cdots, r$, $P_1P_2\cdots P_i$ is normal. A Hall π-subgroup is a π-subgroup whose index is divisible by no prime in π; a Hall π^*-subgroup is a Hall π-subgroup which is also a π^*-group. It is proved that if the finite group G is generated by its subnormal subgroups $G_1, G_2, \cdots, G_n$, and each of $G_1, \cdots, G_n$ possesses a Hall π^*-subgroup, then G possesses a Hall π^*-subgroup. There is a similar theorem in which π is considered in more than one order. In particular, if each of $G_1, G_2, \cdots, G_n$ has a nilpotent Hall π-subgroup, then G has a nilpotent Hall π-subgroup.

Graham Higman (Oxford)

1620:

Wielandt, Helmut. Über die Normalstruktur von mehrfach faktorisierten Gruppen. J. Austral. Math. Soc. **1** (1959/61), 143–146.

Suppose that a finite group G can be expressed in several ways as a product of subgroups: $G=AB=CD=\cdots$. The author seeks relations between the composition factor groups of G and those of $A, B, \cdots$. Main theorem: Let the finite group G contain subgroups $F, G_1, \cdots, G_r$ such that $FG_1=\cdots=FG_r=G$ and such that each composition factor of F is prime to at least one of the n indices $|G:G_i|$. Then every composition factor group of G is isomorphic to a composition factor group of one of the G_i. The main theorem yields the following extension of results of P. Hall [J. London Math. Soc. **12** (1937), 198–200] about Sylow systems in soluble groups. Theorem: Let the order $|G|=a_1a_2\cdots a_r$, where $(a_i, a_j)=1$ for $i\neq j$. G is called an a-group when each of its composition factors is a divisor of one of the a_i. Then the following conditions are all equivalent: (1) G is an a-group; (2) G contains subgroups $G_1, \cdots, G_r$ with indices $|G:G_i|=a_i$ such that whenever $i\neq j$ the meet $D_{ij}=\bigcap_{i\neq k\neq j} G_k$ is an a-group; (3) G contains subgroups $A_1, \cdots, A_r$ with orders $|A_i|=a_i$ such that whenever $i\neq j$ the product $A_iA_j=A_jA_i$ is an a-group. If these conditions hold, the composition factor groups of G are, with the correct multiplicities, those of $A_1, A_2, \cdots, A_r$ taken together; also $|D_{ij}|=|A_iA_j|=a_ia_j$ and $A_1A_2\cdots A_r=G$. The author asks, finally, what permutability conditions on the Sylow subgroups of G are sufficient to guarantee the solubility of G. He gives such conditions when $|G|$ has the form $p^aq^br^cs^d$.

G. E. Wall (Sydney)

1621:

Kalužnin, L. A. Über eine Galois-Beziehung in der Gruppentheorie. Ukrain. Mat. Ž. **11** (1959), 38–51. (Russian. German summary)

Let (*) $G=G_0>G_1>\cdots>G_{m-1}>G_m=1$ be a finite invariant series of a group G and A the group of all automorphisms of G that induce the identity in each factor-group G_{i-1}/G_i $(i=1, 2, \cdots, m)$. A is called the stability group of the series (*) and is known to be nilpotent of class at most $m-1$ [see Kalužnin, *Ber. Math.-Tagung Berlin*, pp. 164–172, Deutscher Verlag, Berlin, 1953; MR **17**, 456; or P. Hall, Illinois J. Math. **2** (1958), 787–801; MR **21** #4183]. In this paper the author shows that if G is an elementary abelian p-group, the class of A is always precisely $m-1$, whatever the series (*). Conversely, with a given group A of automorphisms of a group G two series can be associated as follows. (i) Let $\underline{G}_1$ be the group generated by all $g^{-1}g^\alpha$, $g\in G$, $\alpha\in A$ (this group is normal in G); and by recurrence $\underline{G}_{i+1}$ the normal closure in G of the group generated by all $g^{-1}g^\alpha$, $g\in\underline{G}_i$, $\alpha\in A$. (ii) Let F_1 be the greatest normal subgroup of G whose elements are fixed under all $\alpha\in A$, and by recurrence F_{i+1} the maximal normal subgroup of G containing F_i and such that its elements are fixed modulo F_i under all $\alpha\in A$. If for a natural number m (as small as possible) we have $\underline{G}_m=1$ or $F_m=G$ (these conditions are equivalent), then G is called A-nilpotent. In this case we set $F_{m-i}=\bar{G}_i$ and have in (*) $\underline{G}_i\leq G_i\leq\bar{G}_i$ for $i=1, 2, \cdots, m$. The series (*) is called closed if $\underline{G}_i=G_i=\bar{G}_i$. The questions arise how to characterize by internal properties those series (*) of a given group G that are closed and those subgroups of the full automorphism group of G that are the stability groups of suitable invariant series (*). In this paper the author treats the case of an elementary abelian group of order p^n. He proves that here every subgroup series (*) of G is closed and that stability groups are precisely those p-groups contained in $\mathrm{GF}(n, p)$ that are normal in, and coincide with the intersection of, all Sylow p-subgroups in which they are contained.

K. A. Hirsch (St. Louis, Mo.)

1622:

Brenner, J. L. The linear homogeneous group. III. Ann. of Math. (2) **71** (1960), 210–223.

[For part II see same Ann. (2) **45** (1944), 100–109; MR **5**, 228.]

Let $\mathrm{GL}_n(R)$ be the group of $n\times n$ invertible matrices with elements in a commutative ring R with unity, and set

$$\mathrm{GL}_n^+(R)=\{X\in\mathrm{GL}_n(R):\det X=1\}.$$

The author obtains an assortment of theorems about these groups, mostly for the special cases where $R=Z$ (rational integers) or $R=Z_t$ $(=Z/(t))$, $t>0$.

Let $Q_{m,n}(R)$ be the normal subgroup of $\mathrm{GL}_n(R)$ generated by $I+me_{12}$, where e_{ij} denotes a matrix with 1 in position (i, j) and zeroes elsewhere, and let $Q^+_{m,n}$ be the analogously defined normal subgroup of GL_n^+. Set

$$N^*_{m,n}=\{X\in\mathrm{GL}_n(R):X\equiv\text{scalar matrix }(\mathrm{mod}\,m)\},$$

$$N_{m,n}=\{X\in\mathrm{GL}_n^+(R):X\equiv I\,(\mathrm{mod}\,m)\}.$$

Typical results are as follows. (1) For $n>2$, every normal subgroup N of $\mathrm{GL}_n(Z)$ satisfies $Q_{m,n}\subset N\subset N^*_{m,n}$ for some m. (2) When $R=Z_t$, $Q_{m,n}=N_{m,n}$ for all n. (3) When $R=Z$, $Q^+_{m,2}=N_{m,2}$ for $m=1, 2, 3, 4, 5$. {The author conjectures that (3) holds for all m, or that at least for each m, $Q^+_{m,2}$ contains $N_{mk,2}$ for some k. Both conjectures have recently been disproved by the reviewer in a note "Normal subgroups of the unimodular group" to appear in Proc. Amer. Math. Soc. The corresponding conjectures for $n>2$ are still unsettled.} (4) For $R=Z[i]$, $Q^+_{1,n}=N_{1,n}=\mathrm{GL}_n^+$ when $n>2$, but $Q^+_{1,2}$ is a proper subgroup of GL_2^+. (5) The groups $Q_{m,n}$ and $N^*_{m,n}$ are characteristic in $\mathrm{GL}_n(Z)$. (6) Let $\mathrm{SL}_2(R)$ be the subgroup of $\mathrm{GL}_2(R)$ generated by the transvections $\{I+\alpha e_{ij}:\alpha\in R, i\neq j\}$. If 6 is a unit in R, then $\mathrm{SL}_2(R)$ coincides with its commutator subgroup.

I. Reiner (Urbana, Ill.)

1623:

Suprunenko, D. A. On real linear nilpotent groups. Mat. Sb. (N.S.) **49 (91)** (1959), 347–352. (Russian)

The author applies his previous results [Mat. Sb. (N.S.) **35 (77)** (1954), 501–512; Izv. Akad. Nauk SSSR. Ser. Mat. **19** (1955), 273–274; MR **16**, 793; **17**, 456] on irreducible nilpotent subgroups of the linear group $GL(n, P)$ over an algebraically closed field P to obtain corresponding results for the linear group $GL(n, R)$, where R is the field of real numbers. When n is odd, then $GL(n, R)$ has no irreducible nilpotent subgroups. When n is even, then $GL(n, R)$ contains irreducible nilpotent subgroups of any preassigned class of nilpotency. The maximal irreducible nilpotent subgroups of $GL(n, R)$ of a given class of nilpotency fall into a finite number of conjugate sets. When n is not a power of 2, then the problem of classifying the maximal irreducible nilpotent subgroups of $GL(n, R)$ of a given class of nilpotency reduces to the same problem for $GL(n/2, C)$, where C is the field of complex numbers.

As an illustration the author gives a complete account of the case of real 2×2 matrix groups. He shows that $GL(2, R)$ contains, apart from conjugacy, only one maximal irreducible nilpotent subgroup of a given class of nilpotency c. For $c=1$, this group consists of the matrices

$$\rho\begin{pmatrix}\cos\varphi & -\sin\varphi\\ \sin\varphi & \cos\varphi\end{pmatrix}$$

with positive ρ; for $c>1$, it consists of the matrices

$$\rho\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}^s\begin{pmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{pmatrix}^t$$

with positive ρ, integral s, t and $\alpha=2\pi/2^c$.

K. A. Hirsch (St. Louis, Mo.)

1624:

Ribenboim, P. Un théorème de réalisation de groupes réticulés. Pacific J. Math. **10** (1960), 305–308.

The theorem of Lorenzen that a commutative l-group G is order isomorphic to a subdirect sum of a cardinal sum $\sum G_\gamma$ ($\gamma\in I$) of ordered groups G_γ is proven by making use of the lattice of all filets of G. See P. Lorenzen [Math. Z. **45** (1939), 533–553; MR **1**, 101] and G. Birkhoff [Ann. of Math. (2) **43** (1942), 298–331; MR **4**, 3] for earlier proofs of this result. The author also shows that there exists an o-isomorphism θ of G onto a subdirect sum of $\sum G_\gamma$ with the following properties. (1) $\theta(G)$ is a sublattice of $\sum G_\gamma$. (2) If $\gamma\in I$, $f\in G^+$ and $\gamma\notin\sigma(f)$, then there exists a g in G^+ such that $\gamma\in\sigma(g)$ and $\sigma(g)\cap\sigma(f)=\emptyset$, where $\sigma(g)$ is the set of all γ in I for which $\theta(g)$ has a non-zero γth coordinate. (3) If $\alpha, \beta\in I$, $\alpha\neq\beta$, then there exist f and g in G^+ such that $\alpha\in\sigma(f)$, $\beta\in\sigma(g)$ and $\sigma(f)\cap\sigma(g)=\emptyset$. (4) If $f, g\in G^+$, then $\sigma(f)=\sigma(g)$ if and only if f and g belong to the same filet of G. *P. F. Conrad* (New Orleans, La.)

1625:

Choe, Tae Ho. Erratum: Notes on lattice-ordered groups. Kyungpook Math. J. **2** (1959), 73.

The corrections and simplifications noted in the review [MR **20** #5809] of his article in same J. **1** (1958), 37–42 are acknowledged by the author.

P. F. Conrad (New Orleans, La.)

1626:

Kalman, J. A. Triangle inequality in l-groups. Proc. Amer. Math. Soc. **11** (1960), 395.

Proof that if $|a+b|\leq|a|+|b|$ for all a and b in the additive l-group G, then G is commutative.

1627:

Świerczkowski, S. On cyclically ordered groups. Fund. Math. **47** (1959), 161–166.

A cyclically ordered group G is a group together with a set of ordered triples of distinct elements of G called a cyclic order, with membership denoted by $[x, y, z]$, satisfying: (I) either $[x, y, z]$ or $[z, y, x]$; (II) $[x, y, z]\Rightarrow[y, z, x]$; (III) $[x, y, z]$ and $[y, u, z]\Rightarrow[x, u, z]$; (IV) $[x, y, z]\Rightarrow[uxv, uyv, uzv]$ for $u, v\in G$. The familiar cyclically ordered group of complex numbers of absolute value one is denoted by K. If Γ is a linearly ordered group, the direct product $K\times\Gamma$ becomes a cyclically ordered group if it is provided with the "natural cyclic order", defined in an obvious manner analogous to the lexicographic order. The author proves that if G is any cyclically ordered group, then there exists an ordered group Γ such that $G\subset K\times\Gamma$, and the cyclic order on G is that induced by the natural cyclic order on $K\times\Gamma$. If G is in addition Archimedean, i.e., does not contain elements x, y such that $[e, x^n, y]$ for every positive integer n (where e is the identity), then $G\subset K$ and its cyclic order is induced by the cyclic order on K. *C. W. Kohls* (Rochester, N.Y.)

1628:

Świerczkowski, S. On cyclically ordered intervals of integers. Fund. Math. **47** (1959), 167–172.

[See the preceding review for definitions.] A cyclically ordering relation on the set of integers $\{0, 1, \cdots, N\}$ is a cyclic order on the additive group of integers, except that only triples of elements in $\{0, 1, \cdots, N\}$ are permitted. In the cyclically ordered group K, consider $p_x=\exp(2\pi i\eta x)$, where $x=0, 1, \cdots, N$, and η is a real number such that the p_x are distinct points. A cyclically ordering relation relative to η may be defined by transferring to $\{0, 1, \cdots, N\}$ the cyclic order induced on $\{p_0, p_1, \cdots, p_N\}$ by the cyclic order on K. The author proves that any cyclically ordering relation on $\{0, 1, \cdots, N\}$ coincides with the type just defined, relative to all η in some non-empty open interval.

C. W. Kohls (Rochester, N.Y.)

1629:

Vagner, V. V. Representation of generalized heaps. Ukrain. Mat. Ž. **11** (1959), 231–242. (Russian. English summary)

A function from $K\times K\times K$ to K, whose value at (a, b, c) is denoted $[abc]$, is a semiheap on K if it satisfies the identities $[[k_1k_2k_3]k_4k_5]=[k_1[k_4k_3k_2]k_5]=[k_1k_2[k_3k_4k_5]]$ (note the middle term). Examples of semiheaps are the following. (1) $ab^{-1}c$ or $cb^{-1}a$, derived from a group. (2) If A and B are sets, let $P(A\times B)$ be the set of subsets of $A\times B$, and for $\rho\in P(A\times B)$ let ρ^{-1} be the relation inverse to ρ. Define $[\rho_1\rho_2\rho_3]$ to be $\rho_3\circ\rho_2^{-1}\circ\rho_1$, where $\circ$ is the usual composition of relations. This defines a semiheap on $P(A\times B)$. (3) A semiheap on $P(A\times B)\times P(B\times A)$ is obtained by setting

$$[(\rho_1, \sigma_1)(\rho_2, \sigma_2)(\rho_3, \sigma_3)]=(\rho_3\circ\sigma_2\circ\rho_1, \sigma_1\circ\rho_2\circ\sigma_3).$$

The author develops a semiheap representation theory analogous to the representation of groups by permutation groups. For example, (almost) every semiheap is isomorphic to a subsemiheap of $P(A\times B)\times P(B\times A)$ for suitably constructed A and B.

A semiheap satisfying conditions suggested by commutativity and idempotency is called a generalized heap and is shown to be isomorphic to a subsemiheap of $P(A\times B)$, in fact to a subsemiheap of the partial 1-1 relations on A to B.

This generalization of groups might be contrasted with that given by E. L. Post [Trans. Amer. Math. Soc. 48 (1940), 208–350; MR 2, 128]. *S. Stein* (Davis, Calif.)

TOPOLOGICAL GROUPS AND LIE THEORY

See also 1905.

1630:

Hofmann, Karl Heinrich. Eine Bemerkung über die zentralen Untergruppen in zusammenhängenden topologischen Gruppen. Arch. Math. **9** (1958), 33–38.

The author observes that a totally disconnected normal subgroup of a connected topological group is contained in its center. This leads to an elementary proof of the fact that every torusgroup which is a normal subgroup in a connected group is contained in its center. Several generalizations, some of them to non-associative structures, are discussed. *J. de Groot* (Amsterdam)

1631:

Reiter, H. The convex hull of translates of a function in L^1. J. London Math. Soc. **35** (1960), 5–16.

Let G be a locally compact group and H a closed subgroup such that the homogeneous space of left cosets, G/H, has an invariant measure. Then for $f\in L^1(G)$,

$$\inf\int_G|\sum_{n=1}^{N}c_nf(x\alpha_n)|dx=\int_{G/H}dx'\left|\int_H f(xy)dy\right|,$$

where the infimum is taken over all $\alpha_n\in H$ and over all $c_n\geq 0$, $\sum_1^N c_n=1$, as N ranges over all positive integers. The measures dx, dy above are the left Haar measure on G and H (normalized as usual), and dx' is the invariant measure of the homogeneous space. If in addition, H is a normal subgroup, then one also has

$$\inf\int_G|\sum_{n=1}^{N}c_nf(\alpha_nx)|dx=\int_{G/H}dx'\left|\int_H f(xy)dy\right|$$

with the infimum over the same set as above.

P. Civin (Gainesville, Fla.)

1632:

Yamaguti, Kiyosi. A note on a theorem of N. Jacobson. J. Sci. Hiroshima Univ. Ser. A **22**, 187–190 (1958).

This paper completes the characterization, in terms of operations satisfying identities, of three related types of subspaces of Lie algebras. The "strongest" type is the Lie triple system. The next is the author's general Lie triple system [same J. **21** (1957/58), 155–160; MR **20** #6483]. These systems are imbeddable as complements of subalgebras invariant with respect to the regular representation of the subalgebra in some Lie superalgebra. The weakest type, called the h system by the author, is realizable as a complement of a subalgebra of a Lie algebra. *W. G. Lister* (Oyster Bay, N.Y.)

1633:

Tits, J. Une remarque sur la structure des algèbres de Lie semi-simples complexes. Nederl. Akad. Wetensch. Proc. Ser. A **63** = Indag. Math. **22** (1960), 48–53.

Let L be a Lie algebra over a field K, with a subalgebra M, and let U be a complementary subspace of M in L such that $UU\subseteq M$ and $UM\subseteq U$. Then L is determined by M and U together with a bilinear mapping $\pi: U\otimes U\to M$ and a representation ρ of M in $E(U)$, the Lie algebra of all endomorphisms of U. Consider the particular case where $U=W\otimes V_2$ is the tensor product of a space W and a two-dimensional space, $M=N+E^0(V_2)$ is the direct sum of an algebra N and the algebra of endomorphisms of zero trace of V_2 and $\rho=\tau\otimes\iota$ is the tensor sum [cf. C. Chevalley, *Théorie des groupes de Lie, III*, Hermann, Paris, 1955; MR **16**, 901] of a representation $\tau: N\to E(W)$ and the inclusion $\iota: E^0(V_2)\to E(V_2)$. The expression $L=(N+E^0(V_2))+W\otimes V_2$ is then called a Λ-decomposition of L, with characteristic functions π and ρ. In fact, these functions together with N and W determine L completely. When K has characteristic not two, the mapping π may be replaced by two bilinear functions on W with values in K and N respectively. The author now proves that every semisimple complex Lie algebra L admits a Λ-decomposition in which N is reductive and shows how to deduce the structure of N and the representation τ from the diagram of the simple roots of L [cf. Dynkin, Mat. Sb. (N.S.) **30** (**72**) (1952), 349–462; MR **13**, 904].

P. M. Cohn (Manchester)

1634:

Curtis, Charles W. Representations of Lie algebras of classical type with applications to linear groups. J. Math. Mech. **9** (1960), 307–326.

Let L be a Lie algebra of classical type [Mills and Seligman, J. Math. Mech. **6** (1957), 519–548; MR **19**, 631] over an algebraically closed field Ω of characteristic $\neq 2, 3, 5, 7$. It is shown that L is a restricted Lie algebra [Jacobson, Trans. Amer. Math. Soc. **50** (1941), 15–25; MR **3**, 103] if Ω has prime characteristic; in this case, "module" and "representation" of L will always mean "restricted module" and "restricted representation". The author uses methods analogous to those of Weyl and Harish-Chandra to establish the correspondence between irreducible representations of L and (dominant) integral linear functions on a Cartan subalgebra of L. The component G of the identity in the group of automorphisms of L is a group studied by Chevalley [Tohôku Math. J. (2) **7** (1955), 14–66; MR **17**, 457]. It is now assumed that L has non-degenerate Killing form; then the Lie algebra of G may be identified with L. Each irreducible L-module M becomes the space of an irreducible projective linear representation of G. Relations of equivalence of these L-modules and representations of G are investigated. A group G^* is introduced, G^* being a subdirect product of the images of G under the representations above, and playing the part of a covering group for G. If M is an irreducible L-module such that the representation of L in the tensor algebra $T(M)$ has submodules isomorphic to all irreducible L-modules, then G^* may be realized as a

linear group acting in M. This is the case for most of the classical special linear and symplectic Lie algebras, where interpretations of G^* and of G are given.

G. B. Seligman (New Haven, Conn.)

MISCELLANEOUS TOPOLOGICAL ALGEBRA

1635:

Hunter, R. P. Certain upper semi-continuous decompositions of a semigroup. Duke Math. J. **27** (1960), 283–289.

An I-semigroup denotes a topological semigroup which is a continuum irreducibly connected between a zero and an identity. Making use of a theorem of the reviewer on the existence of I-semigroups in compact connected partially ordered semigroups, the author establishes the following. Let S be a compact connected normal ($xS=Sx$, each $x\in S$) semigroup with identity. Then (1) if each subgroup of S is totally disconnected, then S contains an I-semigroup from 0 to identity; (2) if S is embeddable in the plane, then S contains a local one-parameter semigroup at the identity. Some examples are presented which illustrate that I-semigroups or local one-parameter semigroups need not exist in compact locally connected semigroups.

R. J. Koch (Baton Rouge, La.)

1636:

Wallace, A. D. Remarks on affine semigroups. Bull. Amer. Math. Soc. **66** (1960), 110–112.

An affine semigroup is a topological semigroup which is a convex subset of a linear topological space with the property that $x(ty+(1-t)z)=txy+(1-t)xz$ and $(ty+(1-t)z)x=tyx+(1-t)zx$ for any x, y and $z\in X$ and any t with $0\leq t\leq 1$. The author announces several theorems which are generalizations of known results about affine semigroups.

Haskell Cohen (Baton Rouge, La.)

FUNCTIONS OF REAL VARIABLES

See also 1765.

1637:

Marcus, S. Remarques sur la superposition de deux fonctions réelles. Colloq. Math. **7** (1959), 79–81.

f, g: fonctions réelles, définies sur $\langle 0,1\rangle$, telles que $0\leq f(x)\leq 1$. Fonction de fonction $h: h(x)=g(f(x))$ pour $0\leq x\leq 1$. Théorème 1: (Hypothèses) f jouit de la propriété de Darboux. L'ensemble des points de non-dérivabilité de f est de mesure positive sur chaque sous-intervalle de $\langle 0,1\rangle$. g n'est pas constante sur $\langle a,b\rangle\subset f(\langle 0,1\rangle)$. (Conclusion) h ne peut avoir une dérivée continue partout sur $\langle 0,1\rangle$. D'après un exemple de E. Marczewski il existe une fonction f appliquant biunivoquement la droite numérique $(-\infty,+\infty)$ sur elle-même, partout discontinue, ne satisfaisant pas à la condition de Darboux. Pour $g=f^{-1}$, $h(x)=x$ quel que soit x. Théorème 2: (Hypothèses) f est continue et non constante sur $\langle 0,1\rangle$. g est continue et non dérivable en chaque point de $\langle a,b\rangle=f(\langle 0,1\rangle)$. (Conclusion) h admet des points de non-dérivabilité et n'est pas à variation bornée sur $\langle 0,1\rangle$. {Remarques du rapporteur: (i) A la troisième ligne de la démonstration de Théorème 1, il faut comprendre "constante" avec la restriction "sur $f(\langle 0,1\rangle)$". (ii) L'intervalle $\langle a,b\rangle$ ne semble pas intervenir dans cette démonstration. (iii) L'exemple de E. Marczewski se rapporte au cas où les domaines de définition de f et g sont $(-\infty,+\infty)$.}

Chr. Pauc (Nantes)

1638:

Coles, W. J. A general Wirtinger-type inequality. Duke Math. J. **27** (1960), 133–138.

Let M be a positive integer and $N=2M$. Let k_i $(0\leq i\leq M)$ be 0 or 1 such that $\sum_{i=0}^{M}k_i$ is even. Let $h_i=\sum_{j=0}^{i}k_j$, $p_i=(-1)^{h_i}$, $q_i=(-1)^i p_i$. Let p be a real continuous function on $[a,b]$, $c_i=(1-k_i)a+k_i b$, $d_i=k_{i+1}a+(1-k_{i+1})b$ and $d_i^*=a+b-d_i$. The following theorem is proved. If the differential equation $y^{(N)}-py=0$ has a solution y such that $(-1)^M py\geq 0$ (but not identically zero) on $[a,b]$, $p_i y^{(M-i)}(c_i)\geq 0$ for $1\leq i\leq M$, and $q_i y^{(M+i)}(d_i)\geq 0$ for $0\leq i\leq M-1$, then the inequality $(-1)^M\int_a^b pu^2dx\leq\int_a^b[u^{(M)}]^2dx$ holds for every function u satisfying $u^{(M)}\in L^2$ and $u^{(i)}(d^*_{M-i-1})=0$ $(0\leq i\leq M-1)$. The equality case is also discussed.

Ky Fan (Detroit, Mich.)

1639:

Bellman, Richard. On inequalities for differential operators. Proc. Amer. Math. Soc. **9** (1958), 589–597.

The author deals with inequalities relating the norm of a function and its derivatives. He deals with a variety of such inequalities by elementary methods, but in many cases he obtains results which are not as strong as those previously known.

E. M. Stein (Chicago, Ill.)

1640:

Voronovskaya, E. V. The functional of the first derivative and improvement of a theorem of A. A. Markov. Izv. Akad. Nauk SSSR. Ser. Mat. **23** (1959), 951–962. (Russian)

Let C_n be the space of algebraic polynomials of degree n with the norm $\|P\|=\max_{0\leq x\leq 1}|P(x)|$; then, according to A. Markov, $|P'(x)|\leq n^2\|P\|$, and, according to S. Bernstein, $|P'(x)|\leq n\|P\|(x-x^2)^{-1/2}$. The author denotes by $N_n(\xi)$ the norm of $P'(\xi)$, considered as a linear functional on C_n; she studies the polynomials $P\in C_n$ with $\|P\|=1$, $P'(\xi)=N_n(\xi)$, as well as properties of the function $N_n(\xi)$. In general, $N_n(\xi)$ is smaller than the Bernstein majorant $n(\xi-\xi^2)^{-1/2}$, but coincides with it in $n+1$ points of $[0,1]$.

G. G. Lorentz (Syracuse, N.Y.)

1641:

Hartman, Philip. On functions representable as a difference of convex functions. Pacific J. Math. **9** (1959), 707–713.

The functions described in the title, called d.c. functions, considered over a finite dimensional Euclidean space, are shown to be closed under composition in the following sense. Let D be convex and open or closed in Euclidean n-space, E^n, and let $y(x)=(y^1(x),\cdots,y^m(x))$ give a mapping of D into an open convex set E of E^m such that $y^j(x)$ is d.c. on D $(j=1,\cdots,m)$. If $g(y)$ is d.c. on E then

$F(x)=g(y(x))$ is d.c. on D. As part of the proof it is shown that a function which is locally d.c. on an open or closed convex set D in E^n is d.c. on D.

E. G. Straus (Los Angeles, Calif.)

1642:

Obreškov, N. Sur quelques représentations intégrales de fonctions réelles sur le demi-axe réel. Bŭlgar. Akad. Nauk Izv. Mat. Inst. **3**, no. 1, 3–28 (1958). (Bulgarian. Russian and French summaries)

The author investigates the convolution transform

$$f(x) = \int_0^\infty \Phi(xt)d\omega(t),$$

where

$$\Phi(x) = \int_0^\infty \cdots \int_0^\infty u_1^{\alpha_1}\cdots u_p^{\alpha_p} \times \exp\left(-u_1-\cdots-u_p-\frac{x}{u_1\cdots u_p}\right)du_1\cdots du_p,$$

and establishes inversion and representation formulas. He introduces the operator L defined by $F_1=x^{\alpha_1+1}(x^{-\alpha_1}F')'$, $F_2=x^{\alpha_2+1}(x^{-\alpha_2}F_1)'$, $\cdots$, $LF=F_p=x^{\alpha_p+1}(x^{-\alpha_p}F_{p-1})'$; we have $L\Phi=(-1)^{p-1}\Phi$. Put $f_0(x)=f(x)$, $f_n(x)=Lf_{n-1}(x)$. The condition for f to be represented by the transform with $\omega(t)$ nondecreasing involves the inequalities $(-1)^{n(p-1)}f_n(x)\geq 0$.

R. P. Boas, Jr. (Evanston, Ill.)

1643:

Tagamlicki, Ya. Über die Abel-Gontscharoffschen Entwicklungen der regulär monotonen Funktionen. Bŭlgar. Akad. Nauk Izv. Mat. Inst. **3**, no. 2, 187–204 (1959). (Bulgarian. German and Russian summaries)

The author shows that every regularly monotonic function (i.e., bounded infinitely differentiable function for which $\varepsilon_n f^{(n)}(x)\geq 0$ on $[0, 1]$, $\varepsilon_n=\pm 1$) admits the representation

$$f(x) = \sum_{\nu=0}^{\infty} a_\nu P_\nu(x) + AP(x),$$

where $a_\nu\geq 0$,

$$P_n(x) = \varepsilon_0 \int_{x_0}^{x} \tau_0 dt_1 \int_{x_1}^{t_1} \tau_1 dt_2 \cdots \int_{x_{n-1}}^{t_{n-1}} \tau_{n-1}dt,$$

$x_n=\frac{1}{2}(1-\varepsilon_n\varepsilon_{n+1})$, $\tau_n=1-2x_n$, and $P(x)$ is independent of f, and is absent if and only if $x_\nu\to 0$ or 1. Special cases were obtained by Sendov in the two papers reviewed below.

R. P. Boas, Jr. (Evanston, Ill.)

1644:

Sendov, Blagovest. Sur une espèce de fonctions regulièrement monotones. Bŭlgar. Akad. Nauk Izv. Mat. Inst. **3**, no. 1, 89–104 (1958). (Bulgarian. Russian and French summaries)

The author considers regularly monotonic functions [see the preceding review] with periodic $\{\varepsilon_\nu\}$.

R. P. Boas, Jr. (Evanston, Ill.)

1645:

Sendov, Bl. Über einige Eigenschaften der regulärmonotonen Funktionen. Bŭlgar. Akad. Nauk Izv. Mat. Inst. **3**, no. 2, 63–80 (1959). (Bulgarian. Russian and German summaries)

The author extends the results of the paper reviewed above to the case where a sequence $1, -1, -1, \cdots, -1, -1, 1$ (or its negative) occurs infinitely often in $\{\varepsilon_\nu\}$. Cf. the second preceding review.

R. P. Boas, Jr. (Evanston, Ill.)

MEASURE AND INTEGRATION

See also 1529.

1646:

Takahashi, Shigeru. On the complete convergence of the Riemann sum. Tôhoku Math. J. (2) **11** (1959), 422–424.

Let $f(t)$ $(-\infty<t<\infty)$ denote a function with period 1 and $(L)\int_0^1 f(t)dt=0$. Let $F_k(t,f)=k^{-1}\sum_{\nu=1}^{k} f(t+\nu k^{-1})$ $(k=1, 2, \cdots)$ denote the Riemann sum of $f(t)$. The sequence $F_{n_k}(t,f)$ converges to zero completely if for every $\varepsilon>0$, $\sum |0\leq t\leq 1, |F_{n_k}(t,f)|>\varepsilon)|<+\infty$. The author proves that if $\{n_k\}$ is an increasing sequence of positive integers then there exists a function $f(t)$ such that $f(t)\in L_p(0, 1)$, $1\leq p<\infty$ and $F_{n_k}(t,f)$ does not converge to zero completely. {The author's notation, coupled with typographical errors and missing symbols, makes the argument difficult to follow; presumably the undocumented reference to Jessen's theorem refers to Ann. of Math. (2) **35** (1934), 248–251.}

M. S. Ramanujan (Ann Arbor, Mich.)

1647:

Srinivasan, T. P. On integration theory. Nederl. Akad. Wetensch. Proc. Ser. A **63**=Indag. Math. **22** (1960), 137–144.

An approach to the theory of abstract Stieltjes-Lebesgue integration is presented, similar to the approach in the reviewer's book *An introduction to the theory of integration* [North-Holland, Amsterdam, 1958; MR **20** #3950]. This implies, therefore, that the integral is defined in terms of the "measure" of ordinate sets. By assuming from the outset on that the linear vector lattice L of functions on which the elementary integral is initially defined satisfies Stone's condition: $f\in L\Rightarrow \min(f, 1)\in L$, and by proving first a lemma on the product measure of an abstract measure and Lebesgue measure, the author shows how to simplify to some extent some of the proofs in the cited book (in particular, the proofs of § 17, Theorems 8 and 9).

A. C. Zaanen (Pasadena, Calif.)

1648:

Henstock, R. A new descriptive definition of the Ward integral. J. London Math. Soc. **35** (1960), 43–48.

The Ward integral [A. J. Ward, Math. Z. **41** (1936), 578–604] adapts the Perron procedure for defining an integral to two finite-valued functions f and φ on a closed linear interval $[a, b]$, by defining a major function $M(x)$ to be such that $M(a)=0$, and for each x of $[a, b]$ there exists a $\delta(x)$ such that $M(t)\geq M(x)+f(x)(\varphi(t)-\varphi(x))$ for $0\leq t-x\leq\delta(x)$ and $M(t)\leq M(x)+f(x)(\varphi(x)-\varphi(t))$ for $0\geq t-x\geq-\delta(x)$. A minor function $m(x)$ is the negative of a major function. If $\inf M(b)=\sup m(b)$, finite, then $W\int_a^b f d\varphi$ exists as this common bound. This note considers a class $\mathfrak{R}$ of right intervals, such that for each x of

$[a, b)$ there exists $h(x)$ such that if $0<h\leq h(x)$, then $x, x+h$ belongs to $\mathfrak{R}$, and similarly a class $\mathfrak{L}$ of left intervals. Classes $\mathfrak{R}$, $\mathfrak{L}$ and functions f, φ, H on $[a, b]$ determine $c(u, v)$ defined on intervals in $\mathfrak{R}\cup\mathfrak{L}$ as $|H(v)-H(u)-f(u)(\varphi(u)-\varphi(v))|$ if (u, v) is in $\mathfrak{R}$, as $|H(v)-H(u)-f(v)(\varphi(u)-\varphi(v))|$ if (u, v) is in $\mathfrak{L}$, and the larger of these values if (u, v) belongs to both $\mathfrak{R}$ and $\mathfrak{L}$. The function $H(x)$ is variationally equivalent to (f, φ) if for every $\varepsilon>0$, there exist two families of intervals $\mathfrak{R}_1$ and $\mathfrak{L}_1$ such that $\sum_i c(x_i, x_{i+1})<\varepsilon$ for all subdivisions $a=x_0<x_1<\cdots<x_n=b$ consisting of intervals of $\mathfrak{R}_1\cup\mathfrak{L}_1$. There exists a function H variationally equivalent to (f, φ) on $[a, b]$ if and only if $W\int_a^b f d\varphi$ exists, and then $H(b)-H(a)=W\int_a^b f d\varphi$. There are indicated possible generalizations to normed linear spaces.

T. H. Hildebrandt (Providence, R.I.)

1649:

Bouligand, G. Sur les transformations conservant les volumes. Ann. Mat. Pura Appl. (4) **48** (1959), 353–359.

Résumé de l'auteur: "Aspects divers de la question. Problèmes à solution unique. Méthodes de fibrage. Atteintes possibles à la dimension d'ensembles. Remarques cinématiques."

1650:

Mešalkin, L. D. A case of isomorphism of Bernoulli schemes. Dokl. Akad. Nauk SSSR **128** (1959), 41–44. (Russian)

Let A be a finite probability space with points $1, 2, \cdots, k$ having probabilities $q_1, \cdots, q_k$ respectively. The set A^∞ of infinite sequences $\{x_n\}$ $(n=0, \pm 1, \pm 2, \cdots)$ of members of A, equipped with the product measure and the shift transformation $T\{x_n\}=\{x_{n+1}\}$, is called a Bernoulli scheme.

It was shown by Kolmogorov [Dokl. Akad. Nauk SSSR (N.S.) **119** (1958), 861–864; **124** (1959), 754–755; MR **21** #2035a, #2035b; see also Sinaĭ, ibid. **124** (1959), 768–771; MR **21** #2036a] that if two Bernoulli schemes $\langle B^\infty, T\rangle$ and $\langle C^\infty, T\rangle$ are isomorphic "mod 0" (up to sets of measure 0) then B and C have the same "entropy" $-\sum q_i \log q_i$. This paper proves that if, for some fixed integer $p>0$, spaces B and C are each obtained from A by amalgamating one or more sets of p, or p^2, or p^3, $\cdots$ points with the same probability, and B and C have the same entropy, $\langle B^\infty, T\rangle$ and $\langle C^\infty, T\rangle$ are actually isomorphic mod 0. In particular, if B and C have the same entropy and all probabilities of points in them are inverse powers of p, we have isomorphism, e.g., for $B=(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$ and $C=(\frac{1}{2}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8})$.

{The part of the proof beginning with the definition of "l-connected pairs" seems to the reviewer to be unclearly exposited. It appears that this definition must be intertwined with the next one, especially to include the second half of condition "(c2)", and with part of the proof of the Lemma, so that it only makes some sense almost everywhere. However, once the construction is understood it is clear that the sets constructed behave as desired under translation by T.}

R. M. Dudley (Princeton, N.J.)

1651:

Alfsen, Erik M. A remark concerning the article: "On a general theory of integration based on order." Math. Scand. **7** (1959), 106–108.

In the review of the previous paper [Math. Scand. **6** (1958), 67–79; MR **21** #709], the reviewer pointed out an error and suggested a correction. In this paper, the author attempts a correction of the paper cited by the same way.

H. Nakano (Kingston, Ont.)

1652:

Vinti, Calogero. Sopra una classe di funzionali che approssimano l'area di una superficie. Ann. Mat. Pura Appl. (4) **48** (1959), 237–255.

Riassunto dell'autore: "Si introducono per via assiomatica, per le superficie continue $z=f(x, y)$, dei funzionali che convergono in area. Si fa vedere poi che non soltanto le medie integrali del primo ordine già usate dal Radó e i polinomi di Stieltjes-Tonelli (e non più 'successioni' di essi) rientrano in tale categoria di funzionali, ma anche, tra l'altro, le medie integrali d'ordine r, qualunque sia $r\neq 0$."

W. H. Fleming (Providence, R.I.)

1653:

Williams, R. F. Lebesgue area of maps from Hausdorff spaces. Acta Math. **102** (1959), 33–46.

Let X be a compact m-dimensional Hausdorff space and f a map from X into Euclidean n-space. Using nerves of open coverings of X two functionals $L_m{}^p$ and $L_m{}^*$ are introduced and compared with the Lebesgue m-dimensional area L_m of the surface represented by f. From the definitions it follows that $L_m{}^p\leq L_m{}^*$. Moreover, $L_m{}^*\leq L_m$ if X is finitely triangulable. If X is a compact 2-manifold it is shown that $L_2{}^*=L_2$. For light maps f it is proved that $L_n{}^*$ does not exceed the lower Riemann integral of the crude multiplicity M_f. An example is given of a light map f from a finitely triangulable space X onto the unit square for which L_2 exceeds both $L_2{}^*$ and the Riemann integral of M_f.

P. V. Reichelderfer (Columbus, Ohio)

FUNCTIONS OF COMPLEX VARIABLES

See also 1587, 1695, 1766, 1774, 1792.

1654:

Teodorescu, N. Romanian and Soviet researches in the theory of areolar derivatives and (α) monogenic functions. Acad. R. P. Romîne. An. Romîno-Soviet. Ser. Mat.-Fiz. (3) **12** (1958), no. 2 (25), 5–19. (Romanian)

Survey article, without bibliography.

1655:

Ozawa, Mitsuru. On extremal quasiconformal mapping. Kôdai Math. Sem. Rep. **11** (1959), 109–123.

The author states some lemmas and theorems on uniformly convergent sequences of quasi-conformal mappings including proofs and applications to extremal quasiconformal mappings of hyperelliptic Riemann surfaces. The following statement may be considered as typical: Let $\{S_\nu\}$ be a uniformly convergent sequence of quasiconformal mappings to the identity mapping of a domain D onto itself; then a suitable subsequence of the complex partial derivatives $\partial S_\nu/\partial z$ and $\partial S_\nu/\partial\bar{z}$ will converge to 1 and 0 respectively a.e.

{Unfortunately these lemmas and theorems, as the preceding one, are not correct, as is seen by the following example: Let $S_\nu = u_\nu + iv_\nu$ (u_ν, v_ν real) be defined by

$$u_\nu(x, y) = \frac{n}{2^\nu} + \frac{1}{2}\left(x - \frac{n}{2^\nu}\right) \quad \text{for } \frac{n}{2^\nu} \leq x \leq \frac{3n+2}{3\cdot 2^\nu},$$

$$= \frac{n+1}{2^\nu} + 2\left(x - \frac{n+1}{2^\nu}\right) \quad \text{for } \frac{3n+2}{3\cdot 2^\nu} \leq x \leq \frac{n+1}{2^\nu},$$

$$v_\nu(x, y) = y \qquad (\nu = 1, 2, 3, \cdots; n = 0, \pm 1, \pm 2, \cdots).$$

The mappings S are quasiconformal, they converge uniformly to the identity mapping, but the dilatation of the S_ν equals 2 a.e., and the complex partial derivatives $\partial S_\nu/\partial z$ and $\partial S_\nu/\partial \bar{z}$ converge nowhere to 1 and 0 respectively.} *A. Pfluger* (Zürich)

1656:

Perry, R. L. The univalent functions of a family. J. London Math. Soc. **35** (1960), 49–62.

The object of this paper is to determine the set of values of t for which the functions of the family $f(z, t) = \sum_{n=1}^{\infty} r_n(t) a_n z^n$, $a_1 = 1$, $0 < t \leq 1$, regular in $|z| < 1$, are univalent on the unit circle U. The $r_n(t)$ are independent functions taking on the values ± 1 and are defined by $r_1(t) = -1$, $0 < t \leq \frac{1}{2}$; $r_1(t) = +1$, $\frac{1}{2} < t \leq 1$; $r_1(t+1) = r_1(t)$ and $r_n(t) = r_1(2^{n-1}t)$.

Let

$$A(f) = \sum_{n=2}^{\infty} n|a_n|, \quad B(f) = \sum_{n=2}^{\infty} n^2|a_n|^2;$$

let $E(t)$ be the set of values of t ($0 < t \leq 1$) for which $f(z, t)$ is univalent in U, $|E(t)|$ its measure, and $CE(t)$ its complement with respect to I, the interval $0 < t \leq 1$. A well known result of univalent functions immediately implies that if $A(f) \leq 1$ then $E(t) = I$. The author proves that if $A(f) > 1$, $a_n \geq 0$, then $|E(t)| < 1$. The function $f(z) = z + \frac{1}{2}qz^2$, $q > 1$, gives a family for which $A(f) = q$ and $CE(t) = I$. The author then constructs a sequence of univalent polynomials with $A(f) = q$ such that $|E(t)| > 0$. The same result is true if $A(f)$ is replaced by $B(f)$. It is also shown that if the a_n are real and $B(f) = \infty$ then $|E(t)| = 0$, while there is a family with $B(f) = \infty$ which contains uncountably many univalent numbers.

Suppose $E(t) = I$. What sort of restriction does this impose upon $A(f)$? The author shows that if $f(z, t)$ is a polynomial of degree three, then $A(f) \leq (6(3)^{1/2} - 9)^{1/2}$. This constant is best possible. A large portion of the paper is taken up in proving the final main result. That is: if $|a_n| \to 0$ as $n \to \infty$ and $\sum_{n=1}^{\infty} |a_n| = \infty$, then $CE(t)$ is dense in I and $|E(t)| < 1$. *W. C. Royster* (Lexington, Ky.)

1657:

Perry, R. L. A theorem on power series whose coefficients have given arguments. J. London Math. Soc. **35** (1960), 172–176.

The author proves that there is no sequence $\eta_1, \eta_2, \cdots$ ($|\eta_n| = 1$, $n = 1, 2, \cdots$) such that every power series $\sum_{n=1}^{\infty} \eta_n a_n z^n$ ($a_n \geq 0$) of positive radius of convergence has its circle of convergence as a natural boundary. For the case that all $\eta_n = \pm 1$ this result was proved by the reviewer [Proc. Amer. Math. Soc. 8 (1957), 443–449; MR **19**, 128]. *W. H. J. Fuchs* (Ithaca, N.Y.)

1658:

R.-Salinas, Baltasar. Disuguaglianze tra limiti e coefficienti dello sviluppo asintotico di una funzione in un angolo. Ann. Mat. Pura Appl. (4) **48** (1959), 147–159.

Riassunto dell'autore: "Si danno disuguaglianze per i coefficienti e limiti dello sviluppo asintotico di una funzione in un angolo e disuguaglianze per i limiti delle derivate di una funzione analitica in un angolo."

1659:

Kakičev, V. A. An integral of Cauchy type for a topological product of two-dimensional analytic surfaces. Dokl. Akad. Nauk SSSR **129** (1959), 1218–1221. (Russian)

A domain D is a topological product of p 2-dimensional analytic manifolds in complex n-space ($n \geq p$), each of which admits a global uniformization by means of a single complex variable. Cauchy's integration formula is generalized to the domain D in a rather obvious way and some applications are indicated.

H. Tornehave (Copenhagen)

1660:

Rodin, Yu. L. Conditions for the solvability of Riemann's and Hilbert's boundary value problems on Riemannian surfaces. Dokl. Akad. Nauk SSSR **129** (1959), 1234–1237. (Russian)

1661:

Kusunoki, Yukio. Theory of Abelian integrals and its applications to conformal mappings. Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. **32** (1959), 235–258.

The author considers "canonical differentials" on an open Riemann surface R, i.e., meromorphic differentials whose integrals are normalized potentials assuming real constant values on ideal boundary components. The class K_0 consists, by definition, of semiexact canonical differentials characterized by vanishing periods along dividing cycles. The following generalization of the Riemann-Roch theorem to open Riemann surfaces is proved.

Let δ be a divisor $P_1^{m_1} \cdots P_r^{m_r}/(Q_1^{n_1} Q_2^{n_2} \cdots)$ on R, $m = \sum m_i < \infty$, and designate by δ_n the restriction of δ to a region R_n of an exhausting sequence $\{R_n\}$, with total order $m - \kappa_n$. Let B_n be the number of linearly independent differentials $\varphi \in K_0$ which are multiples of δ_n and such that $\operatorname{Re} \int \varphi$ is single-valued on $R - R_n$. The number of linearly independent (single-valued) functions in K_0 which are multiples of $1/\delta$ is denoted by A. If p_n is the genus of R_n, then

$$A = B_n + 2(m - \kappa_n - p_n + 1)$$

Consequences are given for surfaces of finite genus. The subclass of meromorphic functions in K_0 is studied. The Riemann-Roch theorem is stated for the class O_{KD} [see review #1663]. Uniqueness and extremal properties of differentials in K_0 are discussed and applications given to conformal mappings of a surface of genus g to a $(g+1)$-sheeted surface. Abel's theorem is generalized for the class of meromorphic functions of the form $\exp \int \varphi$, $\varphi \in K_0$.

An extension of Abel's theorem to open Riemann surfaces has also been given by L. Ahlfors [L. Ahlfors and L. Sario, *Riemann surfaces*, Princeton Univ. Press,

Princeton, N.J., 1960. The Riemann-Roch theorem was similarly generalized by H. Royden [see the next review].

L. Sario (Los Angeles, Calif.)

1662:

Royden, H. L. The Riemann-Roch theorem. Comment. Math. Helv. **34** (1960), 37–51.

The author gives the following generalization of the Riemann-Roch theorem to the class O_{KD} of open Riemann surfaces [see review #1663].

Let G be the space of differentials α on $W \in O_{KD}$ analytic in some neighborhood of the carrier of a given integral divisor δ. Denote by $\mathscr{L}(\delta)$ the set of linear functionals L on G for which $L[\alpha]=0$ for multiples α of δ. Consider the space M of meromorphic functions f on W with a finite number of poles and with a finite Dirichlet integral over the complement of any neighborhood of the poles. Similarly, let D be the space of differentials with a finite number of poles and square integrable over the complement of any neighborhood of the poles. The subspace of differentials in D with vanishing periods along non-dividing cycles is designated by D_{SE}. Let δ_1, δ_2 be disjoint integral divisors. Then $L \in \mathscr{L}(\delta_2)$ is the principal part of an $f \in M$ which is a multiple of δ_1/δ_2 if and only if $L[\alpha]=0$ for all α in D_{SE} which are multiples of δ_1^{-1}.

The theorem remains valid for $W \in O_G$ if D_{SE} is replaced by D. As a corollary, the number of linearly independent $f \in M$ on $W \in O_G$ which are multiples of δ^{-1} exceeds by one the number of linearly independent $L \in \mathscr{L}(\delta)$ such that $L[\alpha]=0$ for all square integrable differentials α.

The above theorem for $W \in O_{KD}$ is then extended to arbitrary Riemann surfaces by imposing upon the functions the further restriction that they are, in essence, real on the ideal boundary or have a constant real part in each boundary component. Generalizations allowing infinite divisors and essential singularities are also indicated.

Extensions of the Riemann-Roch theorem to O_{KD} and to restricted classes of functions on arbitrary Riemann surfaces were independently found by Y. Kusunoki [#1661 above].

L. Sario (Los Angeles, Calif.)

1663:

Royden, H. L. On a class of null-bounded Riemann surfaces. Comment. Math. Helv. **34** (1960), 52–66.

Let K be the class of harmonic functions with vanishing conjugate periods along dividing cycles on an open Riemann surface W. Thus far little is known of the class O_{KD}: it contains O_{KB} and it coincides with O_{KBD} and with S_Q [L. Sario, Trans. Amer. Math. Soc. **73** (1952), 459–470; MR **14**, 863]. Using "Dirichlet principle methods" the author derives several new properties of O_{KD}.

A characterization is first given in terms of the space M of functions g on W with $dg=0$ outside a compact set. A necessary and sufficient condition for $W \in O_{KD}$ is that, for each function f on W with $D(f)<\infty$, there is a function g in M with an arbitrarily small $D(f-g)$.

As a consequence, it is shown that O_{KD} is preserved under quasiconformal mappings while O_{KB} is not. The class O_{AD} for finite genus turns out to be quasiconformally invariant, in interesting contrast with A. Mori's [J. Fac. Sci. Univ. Tokyo, Sect. I **6** (1951), 247–257; MR **13**, 735] result that O_{AD} for infinite genus does not have this property. The invariance has earlier been shown for O_G [A. Pfluger, C. R. Acad. Sci. Paris **227** (1948), 25–26; MR **10**, 28] and for O_{HD} [H. Royden, Proc. Amer. Math. Soc. **4** (1953), 363–370; MR **14**, 864]. The noninvariance has been demonstrated for O_{AB} [A. Mori, loc. cit.]. For O_{HB} the question remains open.

The property of belonging to O_{KD} is shown to be a property of the ideal boundary. For surfaces of finite genus a number of old imbedding theorems are re-established and new ones introduced. A sufficient Ahlfors-type condition is given for $W \in O_{KD}$: the divergence of $\int_0^\infty \Lambda(\rho)^{-1}d\rho$, where now $\Lambda(\rho)$ comes from the complementary component with the longest border.

L. Sario (Los Angeles, Calif.)

1664:

Jurchescu, Martin. Sur le problème du prolongement des surfaces de Riemann. C. R. Acad. Sci. Paris **249** (1959), 988–990.

The capacity $c_\gamma = e^{-k_\gamma}$ of an ideal boundary component γ of an open Riemann surface W can be defined in a conformally invariant manner [Sario, Ann. of Math. (2) **59** (1954), 135–144; MR **15**, 518]. If $c_\gamma=0$, then γ is called weak; if every γ is weak, then the ideal boundary β of W is, by definition, absolutely disconnected. The author defines analogous concepts for bordered Riemann surfaces W_0. The Robin constant k_γ is replaced, in essence, by the extremal distance μ_γ of $\gamma \subset \beta$ from the compact relative boundary of a neighborhood of γ. In terms of μ_γ, several results in the classification theory of bordered Riemann surfaces are then announced without proofs.

For open surfaces the class M_1 is defined by the property that for every subregion of $W \in M_1$ with compact relative boundary ideal boundary components γ have $\mu_\gamma=0$. The subclass $M \subset M_1$ consists of those W that have no γ with a planar neighborhood. The author then announces a solution of the extendability problem of Riemann surfaces: $W \in M_1$ is a necessary and sufficient condition for essential extendability of W. Similarly, $W \in M$ characterizes extendability.

{Proofs of these significant statements are awaited with interest. The following example shows the existence of a W which cannot be essentially extended although β is not absolutely disconnected. Take a hyperbolic $W \in O_{HP}$ with only one boundary component. An essential extension $W^* \supset W$ would contain a disk $D \subset W^* - W$. The principal operator L_0 [L. Ahlfors and L. Sario, *Riemann surfaces*, Princeton Univ. Press, Princeton, N.J., 1960] acting on any nonconstant continuous function $f>0$ on the periphery of D would furnish a positive harmonic function $L_0 f$ on $W^* - D$. This is a contradiction.}

L. Sario (Los Angeles, Calif.)

1665:

Jurchescu, Martin. Sur l'unicité du prolongement des surfaces de Riemann et sur les fonctions de Pompeiu univalentes. C. R. Acad. Sci. Paris **249** (1959), 1026–1027.

The author announces the following properties of the class M_1 of open Riemann surfaces whose boundary components have infinite moduli [cf. preceding review].

(1) A maximal extension (φ, F) of a Riemann surface R is called topologically [conformally] unique if, for any two maximal extensions (φ_1, F_1), (φ_2, F_2), the conformal

self-mapping $\varphi_2 \circ \varphi_1^{-1}$ of R has an extension to a topological [conformal] mapping of F_1 onto F_2. The class of Riemann surfaces with a topologically unique maximal extension coincides with the class M_1.

(2) Let K be the class of harmonic functions u with $\int d\bar{u} = 0$ along dividing cycles, and let H, B, D have their usual meanings. Consider the classes EF ($E = H, K$; $F = B, D$) on a bordered Riemann surface W_0 with compact border B_0. The ideal boundary β_0 of W_0 is EF-removable if every $u \in EF$ with $u = 0$ on B_0, $\int_{B_0} d\bar{u} = 0$, vanishes identically [Sario, *11te Skand. Matematikerkongress*, pp. 229–238, Johan Grundt Tanums Forlags, Oslo, 1952; MR **14**, 863]. Let U be a bordered planar subregion of R with compact border. The author states that every $u \in EF$ on R is EF-extendable on a maximal extension F of R if (a) $R \in M_1$ and (b) every $U \subset R$ has an EF-removable ideal boundary.

(3) A necessary and sufficient condition for a maximal extension of R to be conformally unique is that (a) $R \in M_1$ and (b) every $U \subset R$ has a KD-removable ideal boundary.

(4) For every R there exists another Riemann surface R' with a Pompeiu mapping f from R into R'. Specifically, there is a closed 0-dimensional set $\alpha \subset R$ such that f is analytic on $R - \alpha$. *L. Sario* (Los Angeles, Calif.)

1666:

Rauch, H. E. Weierstrass points, branch points, and moduli of Riemann surfaces. Comm. Pure Appl. Math. **12** (1959), 543–560.

It is known that the closed Riemann surfaces of genus $p \geq 2$ form a complex-analytic manifold (with singularities) of $3p-3$ dimensions, the modulus space, in which the hyperelliptic surfaces form a $2p-1$ dimensional complex-analytic submanifold. Moreover, the Torelli and Teichmüller spaces, successive covering spaces of the modulus space, form complex-analytic manifolds of $3p-3$ dimensions (without singularities) in which the hyperelliptic Torelli or Teichmüller surfaces are again imbedded as $2p-1$ dimensional complex-analytic submanifolds.

It is demonstrated in this paper that "the hyperelliptic surfaces form only one special case of a more general phenomenon, namely, the presence of a whole category of analytic sub-loci of the modulus [Torelli, Teichmüller] space, loci of special surfaces distinguished by the presence of Weierstrass points of particular type". More specifically, the following theorem is proved: The closed Riemann [Torelli, Teichmüller] surfaces of genus p with Weierstrass points whose Weierstrass sequences begin at a fixed $n \leq p$ form a complex-analytic (perhaps disconnected) submanifold of dimension $n+2p-3$ within the modulus [Torelli, Teichmüller] space when $n+1$ is a gap. When $n+1$ is not a gap, they form (at least) a complex-analytic subvariety of dimension $n+2p-4$ within a complex-analytic submanifold of dimension $n+2p-3$.

The theorem is proved by representing the surfaces concerned as n-sheeted coverings of the sphere with an $(n-1)$-fold branch point at ∞. It is then shown that the normalized $n+2p-1$ finite branch points of these representations constitute the required $n+2p-3$ local coordinates. The argument relies upon variational formulae which are computed for the integrals of the first kind, considered as functions of the branch points.

T. Klotz (Los Angeles, Calif.)

1667:

Rauch, H. E. Addendum to "Weierstrass points, branch points, and moduli of Riemann surfaces." Comm. Pure Appl. Math. **13** (1960), 165.

It is noted that in the theorem above "submanifold means the locally homeomorphic image of a manifold by a map of maximum rank", so that, for instance, self-intersections are allowed. *T. Klotz* (Los Angeles, Calif.)

1668:

Garsia, A. M. The calculation of conformal parameters for some imbedded Riemann surfaces. Pacific J. Math. **10** (1960), 121–165.

There exists a known and natural way which assigns to each Riemann surface of genus one a point z in the region $\{z: Iz < 0$, and either $-\frac{1}{2} < Rz < 0$ and $|z| > 1$ or $0 \leq Rz \leq \frac{1}{2}$ and $|z| \geq 1\}$; one sets z equal to the ratio of two principal periods of a suitably chosen abelian differential. Teichmüller [Deutsch. Math. **7** (1944), 309–336; MR **8**, 327] has proved indirectly the existence of such surfaces for which $Rz \neq 0$. The author constructs a class M of surfaces, each subject to the Schottky uniformization and each imbedded in Euclidean three space. From this class he furnishes the first examples of surfaces of genus zero for which $Rz \neq 0$. The parameters involved in the uniformization permit classification of the conformal structure of surfaces in M. The author states that M "may contain all conformal types", but "the question still remains open whether or not every Riemann surface has a conformally equivalent representative in the ordinary three-dimensional space".

W. C. Fox (New Orleans, La.)

1669:

Nakai, Mitsuru. Algebraic criterion on quasiconformal equivalence of Riemann surfaces. Nagoya Math. J. **16** (1960), 157–184.

Let R be a Riemann surface and $M(R)$ the ring of bounded complex valued functions on R which are AC in the sense of Tonelli and have finite Dirichlet integral. Norm $M(R)$ by defining $\|f\| = \sup|f| + [D(f)]^{1/2}$. The author improves on earlier theorems of his to give the following theorems.

Two Riemann surfaces R and R' are quasi-conformally equivalent if and only if the corresponding algebras $M(R)$ and $M(R')$ are algebraically isomorphic.

Two Riemann surfaces R and R' are conformally equivalent if and only if there is a norm preserving isomorphism between $M(R)$ and $M(R')$.

H. L. Royden (Stanford, Calif.)

1670:

Rademacher, Hans. A proof of a theorem on modular functions. Amer. J. Math. **82** (1960), 338–340.

The author gives a new proof of the following theorem. A modular function $\psi(\tau)$ belonging to a modular congruence subgroup modulo N and which is regular and bounded in the half-plane $\Im(\tau) > 0$ is a constant.

He avoids the usual contour integration around the fundamental region of $\psi(\tau)$, which necessitates a study of the structure of this region, in particular of its vertices and cusps.

{Correction: For '$\Re(\tau) > 0$' read '$\Im(\tau) > 0$', both in the theorem and in (1).} *T. Estermann* (London)

1671:

Tsuji, Masatsugu; Shimada, Saburo. Some metrical theorems on Fuchsian groups. Comment. Math. Univ. St. Paul. 8 (1960), 57–62.

Let G be a Fuchsian group of linear transformations $S_n(z)=e^{i\alpha_n}(z-a_n)/(1-\bar{a}_n z)$, $|a_n|<1$. It is assumed that G is finitely generated. The main theorem states that the set $\{e^{i\alpha_n}\}$ is dense on $|z|=1$. The proof uses a result of E. Hopf [Trans. Amer. Math. Soc. **39** (1936), 299–314] {not H. Hopf as erroneously stated}, but the authors acknowledge an earlier unpublished proof by K. Arima.

Various consequences are derived, especially with respect to the angular distribution of the points a_n.

L. Ahlfors (Cambridge, Mass.)

1672:

Tsuji, Masatsugu; Shimada, Saburo. On a Fuchsian group, whose fundamental domain contains no circular arc of $|z|=1$ on its boundary. Comment. Math. Univ. St. Paul. 8 (1960), 77–79.

Essentially the same results as in the paper reviewed above hold under the weaker hypothesis that the limit points of G are dense on $|z|=1$.

L. Ahlfors (Cambridge, Mass.)

1673:

af Hällström, Gunnar. Bemerkungen über Halbvertauschbarkeit ganzer Funktionen. Math. Scand. 7 (1959), 61–70.

Two functions $g(z)$ and $h(z)$ are said to be semipermutable when $g(h(z))\equiv L(h(g(z)))$, where $L(z)$ is a linear function. The author has considered earlier semipermutability of polynomials [Acta Acad. Abo. **21** (1957), no. 2; Math. Japon. **4** (1957), 107–112; MR **18**, 887; **20** #5195]. After establishing some general results about semipermutable functions, the author confines attention to entire functions, and proves that an entire function with a finite exceptional value (in the sense of Picard) is not semipermutable with any entire transcendental function of zero order or non-integer finite order. A non-constant polynomial $h(z)$ is semipermutable with a suitable entire transcendental function $g(z)$ if and only if $h(z)=az+k$, where a is a root of unity. Some more theorems of the same type are given and the paper concludes with consideration of the functions $g(z)$ of the form $Ae^{rz}+B+Ce^{-rz}$, $AC\neq 0$, for which the polynomials $h(z)$ semipermutable with $g(z)$ are given.

S. M. Shah (Lawrence, Kans.)

1674:

Hiong, King-Lai. Sur les fonctions méromorphes en rapport avec leurs primitives. J. Math. Pures Appl. (9) **39** (1960), 1–31.

The author obtains Nevanlinna type inequalities estimating $T(r,f)$ in terms of $\bar{N}(r,f)$, $N(r,1/f^{(-k)})$, $N(r,1/f')$, $N(r,1/f^{(-k+1)})$, p values $N(r,1/(f-a_\mu))$ and q values $N(r,1/(f^{(-k)}-b_\nu))$. Here $f^{(-k)}$ is a primitive of f and it is assumed that $f(z)$, $f^{(-k)}(z)$ are meromorphic in the open z plane. If $a\neq 0$, $b\neq 0$, $a\neq b$, $c\neq\infty$; $f(0)\neq 0$, a, b; $f'(0)\neq 0$; $f^{(-k)}(0)\neq 0$, ∞, c; $f^{(-k+1)}(0)\neq 0$; then

$$(*)\quad T(r,f) < N(r,1/(f-a))+N(r,1/(f-b)) +2N(r,1/(f^{(-k)}-c))-N(r,1/f')+S_{-k}(r,f).$$

The inequality (*) and other inequalities are used to prove results on exceptional values of $f(z)$ when $f^{(-k)}(z)$ has one or more exceptional values.

S. M. Shah (Lawrence, Kans.)

1675:

Hayman, W. K. Picard values of meromorphic functions and their derivatives. Ann. of Math. (2) **70** (1959), 9–42.

The author first proves that, if f is a transcendental meromorphic function in $|z|<R\leqq\infty$, then either f takes every finite value infinitely often or every derivative of f takes every non-zero finite value infinitely often. It is a generalization of Milloux's result who showed it for f regular for $R=\infty$ [H. Milloux, *Les fonctions méromorphes et leurs dérivées*, Hermann, Paris, 1940; MR **7**, 427]. He then discusses the special role of the Picard value zero for derivatives. He conjectures that an integral function f with $f\neq 0$, $f^{(m)}\neq 0$ ($m\geqq 2$) would be reduced to $\exp(az+b)$, which would be a sharper form of a result of Csillag [Math. Ann. **110** (1935), 745–753]. (It has been answered affirmatively in a paper by Tumura [Proc. Phys. Math. Soc. Japan (3) **19** (1937), 29–35], but the author points out that Tumura's proof contains serious gaps.) He proves that it is true for $m=2$. A similar problem is discussed for f meromorphic in $|z|<R\leqq\infty$ and, for example, the following is proved: If f is of finite order in $|z|<\infty$ and such that $f\neq 0$, $f'\neq 0$, $f''\neq 0$, then $f=\exp(az+b)$ or $(az+b)^{-n}$, n being a positive integer. In addition, Picard values of $f'-af^n$ and $f'f^n$ with a positive integer n are discussed.

K. Oikawa (Tokyo)

1676:

Montel, Paul. Éléments exceptionnels des fonctions analytiques. Ann. Sci. École Norm. Sup. (3) **76** (1959), 271–281.

A function $u=f(z)$, meromorphic in a domain D, can be said to define a curve Γ in a projective plane with coordinates z and u. Using this interpretation, the author is led to a number of new questions pertinent to the exceptional values of the meromorphic function $f(z)$.

The tangent of Γ at the point (z,u) is defined by the relation $U-f(z)-f'(z)(Z-z)=0$. Let three points in the (z,u)-plane be given, and let the tangent always meet one of these three points at most a finite number of times. The following counterpart of Picard's Theorem is then proved: Under these conditions, Γ is a straight line, i.e., the function $f(z)$ is a linear polynomial. The result is generalized to the case in which, instead of the tangent, an osculating parabola of degree $p\geqq 1$ is considered.

A point is called tangentially exceptional for $f(z)$ if no tangent of Γ passes through it. For functions $f(z)$ with three prescribed tangentially exceptional values in D, theorems corresponding to those of Schottky and Landau are derived. Some of the results are also extended to functions of two complex variables.

O. Lehto (Helsinki)

1677:

af Hällström, Gunnar. Wertverteilungssätze pseudomeromorpher Funktionen. Acta Acad. Abo. **21**, no. 9, 23 pp. (1958).

Let $w=w(z)$ be an interior transformation (in the sense of Stoïlow) in a general plane domain B. Assume that

$w=w(z)$ of $1/w(z)$ belongs to C^1 in B and the dilatation is bounded on every compact subset of B. Then, $w=w(z)$ is called a pseudo-meromorphic function in B. The purpose of this paper is to extend the author's theory of meromorphic functions in a general plane domain [same Acta **12** (1940), no. 8; MR **2**, 275], which is an extension of Nevanlinna's theory of meromorphic functions in $|z|<R\leq+\infty$, to the case of pseudo-meromorphic functions. The first and second main theorems and the defect-theorem for pseudo-meromorphic functions are obtained both in integrated and unintegrated forms. To obtain the second main theorem, the author uses Dufresnoy's extension of Ahlfors' theory of covering surfaces. Comparisons with meromorphic functions are discussed and sharp estimates of the remainder terms in the first and second main theorems are obtained. *K. Noshiro* (Nagoya)

1678:

Mioduszewski, J. On the necessary and sufficient conditions for the analytic function to be univalent or p-valent in the usual and in the generalized sense. Ann. Polon. Math. **7** (1960), 127–133.

Let $f(z)=z^{-1}+b_1z+\cdots$ in $|z|<1$. Using necessary and sufficient conditions for univalence of $f(z)$ in $|z|<1$ given by Golusin [Mat. Sb. (N.S.) **8 (50)** (1940), 277–284; MR **2**, 185] and Wolibner [Colloq. Math. **2** (1951/52), 249–253; MR **14**, 35] the author obtains a system of inequalities involving the b_n which includes the Bieberbach area principle as a special case. The author also obtains necessary and sufficient conditions in order that a bounded function $f(z)$ regular in $|z|<1$ be (a) p-valent, (b) areal mean p-valent, and (c) circumferentially mean p-valent. {Some of the arguments lack clarity.}

W. C. Royster (Lexington, Ky.)

1679:

Bazilevič, I. E. On an estimate of the mean modulus in a class of bounded univalent functions. Mat. Sb. (N.S.) **48 (90)** (1959), 93–104. (Russian)

Let S_M be the class of functions $f(z)=z+\sum_{n=2}^{\infty}c_nz^n$ that are regular and univalent with $|f(z)|<M$ in $|z|<1$. Set $\tau=1/M$ and let

$$f^*(z)=\frac{(1+z)^2-2\tau z-(1+z)((1+z)^2-4\tau z)^{1/2}}{2\tau z}.$$

It is known that $f^*(z)\in S_M$ and maps $|z|<1$ onto $|w|<M$ except for a suitable slit along the positive real axis. Let $D(r)$ be the image of $|z|\leq r<1$ under $f(z)$ and let $l(r, x, M)$ be the measure of the intersection of the circle $|w|=x$ with $D(r)$. The author obtains the sharp upper bound for $l(r, x, M)$ for $\frac{1}{2}\leq r<1$ and $x\geq 8r$, and shows that equality is attained for $f^*(z)$. The result is used to prove that in the class S_M

$$J(f)\equiv\int_0^r\rho d\rho\int_0^{2\pi}|f(\rho e^{i\theta})|^{\alpha}|f'(\rho e^{i\theta})|^2d\theta$$

does not exceed $J(f^*)+c(r,\alpha)$, where $c(r,\alpha)$ is independent of f and M and tends to zero as $r\to1$. Similar upper bounds are obtained for $\int_0^{2\pi}|f(re^{i\varphi})|^2d\varphi$ and $I(r,f)=(2\pi)^{-1}\times\int_0^{2\pi}|f(re^{i\varphi})|d\varphi$, and it is proved that $\lim_{r\to1}I(r,f)\leq 8M^{1/2}/3\pi+c$, where c is an absolute constant, and the constant $8/3\pi$ is best possible. Further the author generalizes some of these results to the subclass of S_M with k-fold symmetry. *A. W. Goodman* (Lexington, Ky.)

1680:

Mioduszewski, J. On certain estimations of coefficients of univalent analytic functions. Ann. Polon. Math. **7** (1960), 135–140.

The author proves the following theorems which are improvements of theorems of Wolibner [Studia Math. **11** (1949), 126–132; MR **12**, 16]. Theorem 1: Let $f(z)$ be analytic and univalent in $|z|>1$ and have the expansion

$$f(z)=z+\sum_{k=1}^{s}b_{n_k}z^{-n_k}+\sum_{k=n_s+1}^{\infty}b_kz^{-k},$$

where $n_s=n$. If $f(z)\neq0$, n odd and if for every set $a_1,\cdots,a_{s-1}$ of non-negative integers we have $n+1\neq\sum_{k=1}^{s-1}a_k(n_{k+1})$, then $|b_n|\leq 2/n+1$. Theorem 2: Let $f(z)$ be univalent and analytic in $|z|<1$ and

$$f(z)=z+\sum_{k=1}^{s}a_{n_k}z^{n_k}+\sum_{k=n_s+1}^{\infty}a_kz^k,$$

where $n_s=n$ and $a_n\neq0$. If for every set of integers described in theorem 1, $n-1\neq\sum_{k=1}^{s-1}a_k(n_k-1)$, then $|a_n|\leq 2/n-1$. Both these results are sharp, as is shown by the functions $f(z)=z(1+z^{-(n+1)})^{2/n+1}$ and $f(z)=z(1+z^{n-1})^{2/n-1}$, respectively. *W. C. Royster* (Lexington, Ky.)

1681:

Zamorski, J. Differential equations for the extremal starlike functions. Ann. Polon. Math. **7** (1960), 279–283.

The author obtains two differential equations satisfied by a function $F(z)=z^{-1}+\sum_0^{\infty}b_nz^n$ starlike for $0<|z|<1$, which is an extremal for a general real functional $E(b_1,\cdots,b_n)$.

One of the differential equations generalizes a result of Hummel [Proc. Amer. Math. Soc. **9** (1958), 82–87; MR **20** #1779]. The proofs are based on an earlier paper of the author [same Ann. **5** (1958/59), 285–291; MR **21** #5023].

W. Kaplan (Ann Arbor, Mich.)

1682:

Clunie, J.; Keogh, F. R. On starlike and convex schlicht functions. J. London Math. Soc. **35** (1960), 229–233.

Let $f(z)=z+\sum_{n=2}^{\infty}a_nz^n$ be univalent in $|z|<1$. The authors prove: (A) if the image region is starlike with respect to the origin and has finite area Δ, then $|a_n|<2\Delta^{1/2}/\pi^{1/2}(n-1)$, for $n\geq2$, and (B) if the image region is bounded and convex then $a_n=o(1/n)$. By specific examples it is shown that in (A) the order is best possible, and in (B) the inequality $|a_n|>C(\alpha)n^{-1-\alpha}$ for all n can occur, for each positive α. Finally the authors give a simple proof that if $\sum_{n=2}^{\infty}n|a_n|<1$, then $f(z)$ is starlike in $|z|<1$.

A. W. Goodman (Lexington, Ky.)

1683:

Keogh, F. R. Some theorems on conformal mapping of bounded star-shaped domains. Proc. London Math. Soc. (3) **9** (1959), 481–491.

Let f map $|z|<1$ conformally onto a domain D which is bounded and star-shaped with respect to the origin and such that $f(0)=0$. Let C_r denote the image of the circle $|z|=r$, $0<r<1$, and let L_r denote the length of C_r. The author proves that $L_r=O(\log 1/(1-r))$. The mapping f has the representation

$$zf'(z)/f(z)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{e^{i\psi}+z}{e^{i\psi}-z}\,dF(\psi),$$

in which F is a monotone increasing function satisfying $F(\pi)-F(-\pi)=2\pi$ and $\int_{-\pi}^{\pi} F(\psi)d\psi=0$. It is shown that for D bounded and star-shaped with respect to the origin, $\lim_{r\to 1}(re^{i\theta})$ exists and is finite, and $F(\theta)$ is continuous and apart from a constant is equal to $\arg \lim_{r\to 1} f(re^{i\theta})$. If F is also absolutely continuous, then $L_r=o(\log 1/(1-r))$. The author then demonstrates a class C of domains (the unit circle with radial slits of equal length emanating from the circumference) for which F is absolutely continuous, and shows that for any real, non-negative function η defined for $0<r<1$, such that $\lim_{r\to 1}\eta(r)=0$, there is a domain of class C such that for r arbitrarily near 1, $L_r>\eta(r)\log(1/(1-r))$.

G. Springer (Lawrence, Kans.)

1684:

Alenicyn, Yu. E. On some estimates for functions regular in a region of finite connectivity. Mat. Sb. (N.S.) **49 (91)** (1959), 181–190. (Russian)

Let B be a domain of finite connectivity without isolated boundary points and not containing ∞. Let H be the class of functions regular in B (and so single-valued in B), bounded by 1 in absolute value, and vanishing at a given point $a\in B$. There is a function $F(z)\in H$ for which $F'(a)=\sup|f'(a)|$ where f runs through H. The author considers the class $R(b_0, b_1)$ of functions $f(z)=b_0+b_1\times(z-a)+\cdots$ such that $\log f(z)$ is regular in B, and proves the exact estimate $\rho_0\leqq\sup|\arg f(z)|$, where z runs through B, f through $R(b_0, b_1)$, and ρ_0 is the largest root of $\pi|b_1/b_0|=4\rho_0 F'(a)\cos(\pi\alpha/2\rho_0)$; $-\pi<\alpha=\arg b_0\leqq\pi$.

A similar estimate is obtained for the class of functions $g(z)=1+b_1(z-a)+\cdots$ for which $\log f(z)$ is regular in B and $|f(z)|<M$.

Extensions are given to functions $b_0+b_k(z-a)^k+\cdots$ for the case that B is an annulus: $q<|z|<1$. The proofs are based on considering the auxiliary function $(f(z))^{\pi/2\rho}$ whose real part is positive for $\rho>\sup|\arg f(z)|$. The author also states a lower bound for $\sup|f(z)|$ where z runs through B, and $f(z)$ runs through all functions regular in B whose power series at a starts $b_0+b_1(z-a)+\cdots$.

W. H. J. Fuchs (Ithaca, N.Y.)

1685:

Newns, W. F. The product of two simple sets effective in a Faber region. Nederl. Akad. Wetensch. Proc. Ser. A **63** = Indag. Math. **22** (1960), 187–191.

This paper deals with the effectiveness of the product set $\{p_n(z)\}$ of two simple absolutely monic sets of polynomials effective in $D_+(C)$, where C is a simple closed curve. [The terminology and notations are those used by the author in Philos. Trans. Roy. Soc. London Ser. A **245** (1953), 429–468; MR **14**, 968.] It had been shown earlier by the reviewer that such a set is never effective unless C is a circle of centre the origin, and a lower bound had been obtained to define the class of functions represented in $D_+(C)$ by the set $\{p_n(z)\}$. This bound is, however, not exact and the author obtains in this paper the exact value of this bound. In fact, let $z\equiv\chi(t)=(c/t)+\phi(t)$, map $|t|=1$ onto C and $0<|t|<1$ onto the exterior domain $E(C)$, so that $|t|=r$ is mapped onto C_r. Write γ for $|c|$, and define α, β by: $\alpha=\max_{|t|=\rho}|\chi(t)|$; $\beta=\inf\{|t|:|\chi(t)|=\gamma\}$. The required bound is furnished in the following theorem (theorem 1). The product of two simple absolutely monic sets, each effective in $D_+(C)$, is effective for $H(C_r)$ in $D_+(C)$ if and only if $r<\gamma/\lambda^{(1)}(\gamma)$, where $\lambda^{(1)}(\gamma)$ is the Cannon function for the outer set in $|z|\leqq\gamma$.

When the outer set satisfies a normalising condition the value of the bound will be independent of the factor sets, as given in the author's second theorem which includes the following result. If $\{p_n^{(1)}(z)\}$ and $\{p_n^{(2)}(z)\}$ are simple absolutely monic sets effective in $D_+(C)$ and if moreover $\{p_n^{(1)}(z)\}$ is $N(b, b)$ in $D_+(\gamma)$ for some b, then the product set will be effective for $H(C_r)$ in $D_+(C)$ if and only if $r<\gamma/\alpha$.

The results obtained in this paper supersede those given by F. R. Barsoum and the reviewer [same Proc. **63** (1960), 333–337; MR **22** #3823]. *M. Nassif* (Assiut)

1686:

Pommiez, Michel. Sur les zéros des restes successifs des séries de Taylor. C. R. Acad. Sci. Paris **250** (1960), 1168–1170.

Let $f(z)=\sum_0^\infty a_k z^k$, with radius of convergence at least 1, and $f_n(z)=\sum_0^\infty a_{n+k}z^k$, the remainder of order n divided by z^n. The author's main theorem is that there is a number $r<\frac{1}{2}$ such that if $|z_n|\leqq r$, the function $f(z)$ admits the expansion $\sum f_n(z_n)P_n(z)$, where P_n is a polynomial of degree n. Hence if $f_n(z_n)=0$ for every n, with $|z_n|\leqq r$, then $f(z)\equiv 0$. The least upper bound C of such r satisfies $0.5\leqq C\leqq 0.5881$. Various further results in the same circle of ideas.

R. P. Boas, Jr. (Evanston, Ill.)

1687:

Leont'ev, A. F. Sequences of linear aggregates formed from solutions of differential equations. Mat. Sb. (N.S.) **48 (90)** (1959), 129–136. (Russian)

A continuation of the article with the same title in Izv. Akad. Nauk SSSR Ser. Mat. **22** (1958), 201–242 [MR **20** #4002].

1688:

Leont'ev, A. F. On properties of sequences of linear aggregates of Jacobi polynomials and their application to the question of the completeness of the system of Jacobi polynomials. Izv. Akad. Nauk SSSR. Ser. Mat. **23** (1959), 565–594. (Russian)

Let $Dy=\sum_0^s Q_j(z)y^{(s-j)}(z)$, where the Q_j's are analytic functions and the $y^{(k)}$'s are derivatives of y. Sufficient conditions are found for y to lie in the domain of the operator M defined by the relation $My=\sum_0^\infty a_k D^k y$, and an integral representation for M is derived. On the basis of this integral representation, conditions are determined which permit one to conclude, for example, the uniform convergence of a sequence $\{f_m\}$, $f_m(z)=\sum_1^{p_m} P_{r_k}^{(\alpha,\beta)}(z)$ $(m=1, 2, \cdots)$, inside an ellipse with foci at ± 1 from its uniform convergence in a region containing an arc of the ellipse. (The $P_{r_k}^{(\alpha,\beta)}$'s are Jacobi polynomials.) This result enables the author to establish the completeness of a system $\{P_{r_k}^{(\alpha,\beta)}\}$ of Jacobi polynomials on an arc of the complex plane in analogy with his proof [Dokl. Akad. Nauk SSSR **121** (1958), 797–800; MR **21** #1396] for Dirichlet polynomials.

N. D. Kazarinoff (Moscow)

1689:

Videnskiĭ, V. S. On polynomials with least deviation from zero whose coefficients satisfy a given linear relation. Dokl. Akad. Nauk SSSR **126** (1959), 248–250. (Russian)

Désignons par $\{P_n\}$ les polynômes $P_n(z)=\sum_{k=0}^{n} p_k z^k$ satisfaisant aux conditions $\sum_0^n \alpha_k p_k = 1$, $\sum_0^n |\alpha_k| \neq 0$, α_k complexes, et par K un compact dans le plan complexe. L'auteur examine les propriétés du P_n dont la déviation de zéro sur K ($\equiv D_K(P_n)$) est la plus petite. D'après L. G. Schnirelmann [Izv. Akad. Nauk SSSR. Ser. Mat. **1** (1938), 53–60] il existe dans K un système de points $E_K \equiv \{z_\nu\}_1^m$, $m \leq 2n+1$, tel que le P_n dont $D_K(P_n)$ est la plus petite sur K possède la même propriété sur E_K. On appelle un ensemble $E_K \equiv \{z_\nu\}_1^m$, $(1 \leq m \leq 2n)$ caractéristique, si pour tout sous-ensemble $E_K' \subset E_K$ il existe un P_n tel que $D_{E_K'} < D_{E_K}$. L'auteur donne alors des conditions nécessaires et suffisantes que doivent satisfaire $\{P_n\}$ pour que $D_{E_K}(P_n)$ soit le plus petite, ainsi que les propriétés de ces polynômes $\{P_n\}$. *M. Tomić* (Belgrade)

1690:

Brudnyĭ, Yu. A. Approximation by entire functions on the exterior of an interval and on a semi-axis. Izv. Akad. Nauk. SSSR. Ser. Mat. **23** (1959), 595–612. (Russian)

An abstract appeared in Dokl. Akad. Nauk SSSR **124** (1959), 739–742 [MR **21** #255].

R. P. Boas, Jr. (Evanston, Ill.)

1691:

Kakurai, Shigeki. The Bergman kernel of the complex spheres and Poisson integral in its subdomain. Sci. Rep. Tokyo Kyoiku Daigaku. Sect. A **6**, 256–263 (1959).

A complex sphere consists of those n-dimensional ($n \geq 3$) complex vectors Z which satisfy the inequalities $1-2|Z|^2+|Z'Z|>0$ and $|Z'Z|<1$. The author gives an explicit formula for the Bergman kernel of a complex sphere [see also K. Morita, same Rep. **5** (1956), 190–212; MR **19**, 541], describes the behaviour of the kernel function near the part of the boundary given by $|Z|=|Z'Z|=1$, and proves a generalization of Poisson's formula in a special subdomain of the complex sphere.

H. Röhrl (Minneapolis, Minn.)

SPECIAL FUNCTIONS

See also 1784, 2229.

1692:

van der Pol, Balth. On series of the reciprocals of the Jacobian theta functions. Nederl. Akad. Wetensch. Proc. Ser. A **63** = Indag. Math. **22** (1960), 107–114.

Jacobi defined elliptic functions as ratios of theta functions; Weierstrass defined them, essentially, as second logarithmic derivatives of theta functions. In the present paper, a third definition is given, still based on theta functions, but essentially distinct from the Jacobi, or the Weierstrass approach. Let $\alpha=\alpha(v,\tau)=\exp\{-2\pi i(v+\frac{1}{2}\tau)\}$, $\beta=\beta(n,v,\tau)=\exp(-\pi n^2\tau-2\pi i n v)$, and let m, l stand for integers. By iterating $\theta_3(v+\tau)=\alpha\theta_3(v)$, one obtains $\theta_3(v+n\tau)=\theta_3(v)\beta^{-1}$, or, for $v \neq m+\frac{1}{2}+(l+\frac{1}{2})\tau$,

$$\theta_3(v)\{\theta_3(v+n\tau)\}^{-1} = \beta(n,v,\tau).$$

Summing over n, observing that $\sum_n \beta = \theta_3(v)$ and cancelling $\theta_3(v)$, one obtains $\sum_{-\infty}^{\infty}\{\theta_3(v+n\tau,\tau)\}^{-1}=1$. Similarly, for $v \neq$ roots of denominators, $\sum_{-\infty}^{\infty}\{\theta_0(v+n\tau,\tau)\}^{-1}=1$,

$$\theta_1(v)\sum_{-\infty}^{\infty}\{\theta_1(v+n\tau,\tau)\}^{-1} = \theta_0(v),$$

$$\theta_2(v)\sum_{-\infty}^{\infty}\{\theta_2(v+n\tau,\tau)\}^{-1} = \theta_3(v).$$

Setting $u=\pi\theta_3^2 v$, the last two relations yield

$$\sum_{-\infty}^{\infty}\{\theta_1(v+n\tau,\tau)\}^{-1} = k^{-1/2}/\mathrm{sn}\,u,$$

$$\sum_{-\infty}^{\infty}\{\theta_2(v+n\tau,\tau)\}^{-1} = k^{-1/2}\,\mathrm{dn}\,u/\mathrm{cn}\,u,$$

respectively. Many other similar relations are obtained by the same method, of which the following three may be quoted as characteristic:

$$\sum_{-\infty}^{\infty}\theta_0(v+n\tau)\{\theta_1(v+n\tau)\}^{-2} = k^{-1}\,\mathrm{sn}^{-2}\,u = (2\omega_1/\pi\theta_2\theta_3)^2(\wp(z)-e_3);$$

$$\int_{s_0-\infty\tau}^{s_0+\infty\tau}\frac{dv}{\theta_3(v,\tau)} = \tau \quad \text{if} \quad s_0 \not\equiv \tfrac{1}{2} \pmod 1;$$

$$\int_{-\infty}^{\infty}\frac{ds}{\theta_3(s_0+s\tau,\tau)} = 1 \quad \text{if} \quad s_0 \not\equiv 0 \pmod 1.$$

E. Grosswald (Princeton, N.J.)

1693:

Lavoine, Jean. Sur les fonctions factorielles d'Étienne Halphen. C. R. Acad. Sci. Paris **250** (1960), 439–441.

$\mathrm{ef}(\nu;x)$ is the finite part of the integral $\int_0^\infty 2t^{\nu-1}e^{xt-t^2}dt$. The author shows that $\mathrm{ef}(\nu;-x) \leq \Gamma(\nu/2)$ for $x \geq 0$ and $\nu>0$, and derives formulas for the derivatives and integrals of ef, including those of fractional order. The Taylor expansion of $\mathrm{ef}(\nu;x)$ as a function of x follows, and it gives rise to multiplication and addition formulas. The connection of $\mathrm{ef}(-m,x)$ with the Hermite polynomial of degree m is derived. [See the following review.]

A. Erdélyi (Pasadena, Calif.)

1694:

Lavoine, Jean. Sur les fonctions factorielles d'Étienne Halphen. C. R. Acad. Sci. Paris **250** (1960), 648–650.

[See also the preceding review.] Here the author expresses $\mathrm{ef}(\nu;x)$ in terms of parabolic cylinder functions. He also defines $\mathrm{Cf}(\nu;x)$ and $\mathrm{Sf}(\nu;x)$ as, respectively, the even and odd parts of $\mathrm{ef}(\nu;x)$ and gives various formulas for these functions. {Cf and Sf are multiples of confluent hypergeometric series, and virtually all the results given in the two papers are paraphrases of known results for confluent hypergeometric series or parabolic cylinder functions.}

A. Erdélyi (Pasadena, Calif.)

1695:

Kreyszig, Erwin; Todd, John. The radius of univalence of Bessel functions. I. Illinois J. Math. **4** (1960), 143–149.

The main result is that when $\nu>-1$, the function $z^{1-\nu}J_\nu(z)$ is schlicht in the circle $|z| \leq \rho_\nu$ but not in any concentric circle with larger radius, where ρ_ν is the first maximum of the function on the positive real axis. Subsidiary results are given concerning the behaviour of ρ_ν as a function of ν. *F. W. J. Olver* (Teddington)

1696:

Vodička, Václav. Eine Klasse inhomogener Bessel-Gleichungen. Z. Angew. Math. Phys. **10** (1959), 603–608. (English summary)

Functions that satisfy some special nonhomogeneous forms of Bessel's differential equation are noted here, by adjusting the equation after the function is selected.

R. V. Churchill (Ann Arbor, Mich.)

1697:

Toscano, Letterio. Funzioni generatrici del prodotto di due polinomi di Legendre o ultrasferici. Boll. Un. Mat. Ital. (3) **14** (1959), 182–189. (English summary)

The writer gives another proof of the formula

$$(*)\quad \sum_{n=0}^{\infty} \frac{n!}{(2\lambda)_n} t^n P_n^{(\lambda)}(\cos\alpha) P_n^{(\lambda)}(\cos\beta) =$$

$$(1-2t\cos(\alpha+\beta)+t^2)^{-\lambda} F\left[\lambda, \lambda; 2\lambda; \frac{4t\sin\alpha\sin\beta}{1-2t\cos(\alpha+\beta)+t^2}\right]$$

which was proved by the reviewer [same Boll. (3) **14** (1959), 6–9; MR **21** #6451]. The proof makes use of Appell's hypergeometric function of two variables F_4. The writer also obtains a formula for

$$\sum_{n=0}^{\infty} (n+\tfrac{1}{2}) t^n P_n(\cos\alpha) P_n(\cos\beta)$$

in terms of the complete elliptic integral of the second kind.

{L. Robin has informed the reviewer that (*) was proved earlier by A. Ossicini [same Boll. (3) **7** (1952), 315–320; MR **14**, 468].}

L. Carlitz (Durham, N.C.)

1698:

Davison, B. Multilayer problems in the multigroup spherical harmonics method. Canad. J. Phys. **37** (1959), 1482–1498.

The solution of the spherical harmonics equations can often be written as a linear combination of solutions of the Helmholtz equation. It is then possible to solve the spherical harmonics equations by computing the coefficients in this linear combination. In carrying out such a procedure the following problem commonly arises. Given all the spherical harmonics moments on a boundary of a region, one is required to determine the expansion coefficients within the region, so as to continue the solution across it. This could be done by inverting a matrix numerically, but the numerical process can be quite time consuming.

In two earlier papers [same J. **35** (1957), 55–63, 576–593; MR **19**, 882, 100] the author has treated this problem in one energy group and in all one-dimensional geometries. He has shown that the required matrix inversion can be carried out analytically, and has derived an expression for the expansion coefficients as explicit functions of the moments. In slab geometry, for example, this expression contains parameters, $g_n^{(s)}$, satisfying the recursion relation

$$(n+1)g_{n+1}^{(s)} + ng_{n-1}^{(s)} + \frac{(2n+1)}{\kappa_s l}\, g_n^{(s)} = \delta_{0n}\frac{c}{\kappa_s l}\, g_0^{(s)} \quad (0 \le n \le N).$$

The quantity κ_s takes on all values for which the above equations can have non-vanishing solutions.

The author's method has now been generalized to treat multigroup problems with isotropic scattering coefficients, $c_{i\to j}$. In the multigroup case and, again, in slab geometry, it is necessary to solve the equations

$$(n+1)g_{n+1}^{(j;s)} + ng_{n-1}^{(j;s)} + \frac{(2n+1)}{\kappa_s l_j}\, g_n^{(j;s)} = \delta_{0n}\sum_i \frac{c_{i\to j}}{\kappa_s l_i}\, g_0^{(j;s)},$$

together with the adjoint set

$$(n+1)g_{n+1}^{(j;s)} + ng_{n-1}^{(j;s)} + \frac{(2n+1)}{\kappa_s l_j}\, g_n^{(j;s)} = \delta_{0n}\sum_i \frac{c_{j\to i} l_i}{\kappa_s l_j^2}\, g_0^{(j;s)}.$$

The author points out that the $g_n^{(j;s)}$ and $g_n^{(j;s)}$ satisfy certain orthogonality relations. With the aid of these relations, it is again possible to derive an explicit expression for the expansion coefficients, in terms of the values of the spherical harmonics moments at a boundary.

Similar results are obtained for spherical and cylindrical geometries. In cylindrical geometry the analysis is complicated somewhat by the presence of associated Legendre functions, P_l^m, $m \neq 0$.

E. M. Gelbard (Pittsburgh, Pa.)

1699:

Shail, R. Two centred expansion of the retarded Green's function. Mathematika **6** (1959), 108–117.

The problem of expanding the Coulomb Green's function $1/r_{12}$ in terms of two centers arises naturally in expressing the static interaction energy of two charge distributions. A retarded Green's function $\exp(ikr_{12})/r_{12}$ similarly arises in discussing the interaction energy for electromagnetic radiation. The Coulomb Green's function has previously been expanded in two-centered form in terms of associated Legendre functions and trigonometric functions by Carlson and Rushbrooke [Proc. Cambridge Philos. Soc. **46** (1950), 626–633; MR **12**, 259]. The analogous problem is here solved for the retarded Green's function. Two forms of solution are given: one in terms of Hankel functions, the other in terms of associated Legendre functions.

D. E. Spencer (Storrs, Conn.)

1700:

Meulenbeld, B. On derivatives of the generalized Legendre functions. Nederl. Akad. Wetensch. Proc. Ser. A **63** = Indag. Math. **22** (1960), 200–206.

The author first expresses the first and second derivatives of $P_k^{ms}(z)$ and $Q_k^{ms}(z)$ with respect to z in terms of these functions of degrees k and $k-1$. He then discusses the derivatives with respect to k, m, n. The derivative of $P_k^{ms}(z)$ with respect to k is treated in full detail, giving formulas in the various cases according as $k \pm \frac{1}{2}(m-n)$ is or is not an integer; the other derivatives are treated more briefly.

A. Erdélyi (Pasadena, Calif.)

1701:

Levelt, A. H. M. On a formula of C. S. Meijer. Nederl. Akad. Wetensch. Proc. Ser. A **63** = Indag. Math. **22** (1960), 102–105.

C. S. Meijer [same Proc. **56** (1953), 349–357; MR **15**, 422] has expanded a hypergeometric function with variable λ, ξ into a series of hypergeometric functions with variable ξ multiplied by powers of ξ and hypergeometric polynomials of λ. The author proves Meijer's result by expanding all hypergeometric functions occurring in it into power series, and summing the resulting

triple series by changing the orders of summation. He also gives an analogous result for basic hypergeometric series.

A. Erdélyi (Pasadena, Calif.)

1702:

Shukla, H. S. Certain theorems of Cayley and Orr type for bilateral hypergeometric series. Quart. J. Math. Oxford Ser. (2) **10** (1959), 48–59.

This paper gives a number of theorems which generalize the theorems of Cayley and Orr in terms of bilateral series. A typical example is Theorem IV. If

$$\frac{(1-E)\Gamma(1+E-D)\Gamma(D-E)}{\Gamma(2+B-D)\Gamma(2+K-A-D)\Gamma(D-1-K-B)\Gamma(D-1-K)}$$

$$\times {}_1H_1\left[\begin{matrix} D-1-K+A \\ D \end{matrix}; z\right] {}_2H_2\left[\begin{matrix} 2+K+B-D;\ 2+K-D; \\ 2-D \qquad ;\ 2+B-D; \end{matrix} z\right]$$

$$+\text{idem}(D;E) = \sum_{n=-\infty}^{\infty} a_n z^n,$$

then

$$\frac{\Gamma(D)\Gamma(E)\Gamma(2-D)\Gamma(2-E)}{\Gamma(1+B)\Gamma(1+K-A)\Gamma(-K-B)\Gamma(-K)}$$

$$\times {}_2F_1\left[\begin{matrix} \tfrac{1}{2}(1+K+B),\ \tfrac{1}{2}(2+K+B); \\ 1+B \end{matrix} z\right]$$

$$\times {}_2F_1\left[\begin{matrix} \tfrac{1}{2}(A-K),\ \tfrac{1}{2}(1+A-K); \\ 1+A \end{matrix} z\right]$$

$$= \sum_{n=0}^{\infty} \frac{(1+A+B+n)_n}{2^{2n}(1+A)_n} a_n z^n,$$

where $|z|=1$. If $D=1$, we have the corollary: If

$$(1-z)^{K-A} {}_2F_1\left[\begin{matrix} 1+K+B,\ 1+K; \\ 1+B \end{matrix} z\right] = \sum_{n=0}^{\infty} a_n z^n,$$

then

$${}_2F_1\left[\begin{matrix} \tfrac{1}{2}(1+K+B),\ \tfrac{1}{2}(2+K+B); \\ 1+B \end{matrix} z\right]$$

$$\times {}_2F_1\left[\begin{matrix} \tfrac{1}{2}(A-K),\ \tfrac{1}{2}(1+A-K); \\ 1+A \end{matrix} z\right]$$

$$= \sum_{n=0}^{\infty} \frac{(1+A+B+n)_n}{2^{2n}(1+A)_n} a_n z^n,$$

where $|z|<1$.

The author does not show how it is that he can deduce a theorem which holds in $|z|<1$ from one which holds only on $|z|=1$, and the whole problem of convergence is ignored.

L. J. Slater (Cambridge, England)

1703:

Inkeri, K. The real roots of Bernoulli polynomials. Ann. Univ. Turku. Ser. A I **37** (1959), 20 pp.

The roots of the Bernoulli polynomial $B_n(X)$ in the unit interval have been discussed by Lense [*Reihenentwicklungen in der mathematischen Physik*, de Gruyter, Berlin, 1953; MR **14**, 979] and the reviewer [Amer. Math. Monthly **47** (1940), 533–538; MR **2**, 43] independently. The present paper derives their results without the use of Fourier series and obtains further information concerning the roots outside the unit interval. A few interesting results may be quoted here. Lense had shown that the root of $B_{2k}(x)$ which is near to $\frac{1}{4}$ approaches $\frac{1}{4}$ monotonically from the left as k increases (although $B_{2k}(\frac{1}{4})$ tends rapidly to infinity). The author shows more: that $\frac{1}{4}$ exceeds the root by less than $1/(4^k\pi)$ but more than one quarter of this amount. By the von Staudt theorem it is shown that $B_n(x)$ has no rational root other than the ever-present roots 0, $\frac{1}{2}$, 1 of $B_{2k+1}(x)$. $B_n(x)$ has asymptotically $2n/(e\pi)$ real roots symmetrically situated with respect to $x=\frac{1}{2}$. The $2m$th and $2m+1$st positive roots of $B_{2k}(x)$ tend to $m+\frac{1}{4}$ and $m+\frac{3}{4}$ as $k\to\infty$. Similarly, the roots of $B_{2k+1}(x)$ tend to integers and halves of odd integers. All the real roots of $B_n(x)$ are simple except possibly the greatest (and least); these may be at most double roots.

D. H. Lehmer (Berkeley, Calif.)

1704:

Carlitz, L. Some formulas related to the Rogers-Ramanujan identities. Ann. Mat. Pura Appl. (4) **47** (1959), 243–251.

In a recent note [Math. Nachr. **17** (1958), 23–26; MR **20** #1865] the author proved the identities

$$(1)\qquad \sum_0^{\infty} \frac{q^{n^2-n}t^{2n}}{(q)_n} = \sum_{-\infty}^{0} (-1)^m q^{\frac{1}{2}m(m+1)} I_{2m}(t),$$

$$(2)\qquad \sum_0^{\infty} \frac{q^{n^2}t^{2n+1}}{(q)_n} = \sum_0^{\infty} (-1)^m q^{\frac{1}{2}m(m+1)} I_{2m-1}(t),$$

where $(q)_m=(1-q)(1-q^2)\cdots(1-q^m)$, $(q)_0=1$, and $I_n(t)$ is a q-analog of the Bessel function defined by means of

$$\sum_0^{\infty} (1+q^r xt)(1+q^r x^{-1}t) = \sum_{-\infty}^{\infty} x^n I_n(t).$$

In this paper he uses (1) and (2) to derive two sets of identities of which

$$(3)\qquad \sum_0^{\infty} \frac{q^{m^2}}{(q)_m} = \sum_{-\infty}^{\infty} (-1)^m q^{\frac{1}{2}(5m^2+m)} \prod_1^{\infty} (1-q^n)^{-1},$$

$$(4)\qquad \sum_0^{\infty} \frac{q^{m^2+m}}{(q)_m} = \sum_{-\infty}^{\infty} (-1)^m q^{\frac{1}{2}(5m^2+3m)} \prod_1^{\infty} (1-q^n)^{-1},$$

are special cases. Identities (3) and (4) are related to the Rogers-Ramanujan identities [for references see Hardy, *Ramanujan*, Macmillan, New York, 1940; MR **3**, 71; Chapter 6].

A. L. Whiteman (Princeton, N.J.)

1705:

Kharadzé, A. Quelques remarques sur un mode de généralisation, basé sur l'application des racines de l'unité. Bŭlgar. Akad. Nauk Izv. Mat. Inst. **3**, no. 2, 139–153 (1959). (Bulgarian and Russian summaries)

Dans la première partie l'auteur considère une généralisation des procédés d'orthogonalisation, engendrée par l'intégration sur une étoile symétrique à k rayons (essentiellement identique avec les procédés donnés par Kurt Endl, Math. Z. **65** (1956), 1–6 [MR **17**, 1203]), et obtenue en généralisant les résultats de P. Montel, L. Tchakaloff et J. Favard sur le théorème de moyenne.

Le problème de donner une éstimation pour les valeurs précises dans la formule généralisée de moyenne est résolu pour certaines familles de polynômes réels en considérant les polynômes généralisés de Tchebycheff. Les racines cubiques de l'unité sont utilisées dans la deuxième partie pour donner une interprétation de coordonnées complexes, dans la troisième partie pour établir une notion de "courbure au sens d'Appell" et d'autres résultats concernant des espaces quasi-euclidéen.

K. Endl (Columbus, Ohio)

ORDINARY DIFFERENTIAL EQUATIONS

See also 1638, 1742, 1755, 1756.

1706:

Conti, Roberto. Sulla risoluzione dell'equazione $F(t, x, dx/dt)=0$. Ann. Mat. Pura Appl. (4) **48** (1959), 97–102.

An iterative method is developed for establishing the existence of a solution of the implicit first-order differential equation $F(t, x, dx/dt)=0$, when F is continuous and satisfies a Lipschitz condition

$$0 < \lambda_1 \leq [F(t, x, u') - F(t, x, u'')]/(u' - u'') \leq \lambda_2,$$

without applying the implicit function theorem to transform the differential equation into an equivalent one of the form $dx/dt=\varphi(t, x(t))$, as is customary. The method followed is an adaptation of the elegant device of Tonelli, as follows. One defines for $t_0 \leq t \leq t_0+\delta$ for suitable δ a double sequence of functions $x_n(t)$, $u_n(t)$, setting $x_n(t)=x^0$, $u_n(t)=u^0$ for $t_0 \leq t \leq t_0+\delta/n$, and, for $t > t_0+\delta/n$, $x_n(t)=x^0+\int_{t_0}^{t-\delta/n} g(t, x_n(t), u_n(t))dt$, $u_n(t)=g(t-\delta/n, x_n(t-\delta/n), u_n(t-\delta/n))$, where $g(t, x, u)=u+\mu F(t, x, u)$ for a suitably chosen constant μ. It is then shown, following Peano, that the $x_n(t)$ and $u_n(t)$ form a bounded equicontinuous sequence, and that any convergent subsequence converges to a pair such that $u_\infty(t)=dx_\infty(t)/dt$ which gives a solution $x_\infty(t)$ of the differential equation satisfying the condition $x_\infty(t_0)=u_0$. *G.-C. Rota* (Cambridge, Mass).

1707:

Olech, C.; Opial, Z. Sur une inégalité différentielle. Ann. Polon. Math. **7** (1960), 247–254.

If in the differential equation $y'=f(x, y)$, $f(x, y)$ satisfies the Caratheodory conditions on $0 \leq x \leq h$, $-\infty < y < +\infty$, viz., (1) measurable in x for each y, (2) continuous in y for each x, (3) $|f(x, y)| \leq M(x)$ with $\int_0^h M(x)dx < \infty$, and if $\psi(x; a, b)$ is a maximum solution of the differential equation such that $\psi(a)=b$, then for any absolutely continuous function $\varphi(x)$ such that $\underline{D}_+\varphi(x) \leq f(x, \varphi(x))$ and $\varphi(0) \leq \psi(0)$, it is true that $\varphi(x) \leq \psi(x; \xi, \varphi(\xi))$ for $0 \leq \xi \leq x \leq h$. [See F. Cafiero, Giorn. Mat. Battaglini (4) **77** (1947), 145–163; MR **10**, 194]. A necessary and sufficient condition that the conclusion of this theorem holds is that $\varphi(x)$ be of bounded variation whose singular part is nondecreasing. Sufficient condition that the inequality $\underline{D}_+\varphi(x) \leq f(x, \varphi(x))$ almost everywhere on $[0, h]$ imply $\varphi(x) \leq \psi(x, \xi, \varphi(\xi))$ is that $\varphi(x)$ be continuous and of bounded variation with singular part nondecreasing.

T. H. Hildebrandt (Ann Arbor, Mich.)

1708:

Kropivnic'ka, K. Ĭ. Approximate solution of the equation of forced quasiharmonic oscillations. Dopovidi Akad. Nauk Ukraïn. RSR **1959**, 724–729. (Ukrainian. Russian and English summaries)

Using a method developed by M. Ya. Leonov [Inst. Mašinoved. Avtomat. Nauč. Zap. **2** (1953), no. 1, 5–8; MR **17**, 1208; Dokl. Akad. Nauk Ukrain. SSR **3** (1948), 51; see also Dokl. Akad. Nauk SSSR (N.S.) **62** (1948), 161–162; MR **10**, 251] and variation of constants, the author obtains the general solution of the equation (1) $\ddot{x}+2\gamma(\sigma)\dot{x}+x=S(\sigma)$, $\cdot=d/d\sigma$, where γ is a given periodic damping coefficient and S is a given forcing term. For the case $|\gamma(\sigma)| \ll 1$, an approximate solution of (1) is given in a form convenient for computations (involving only trigonometric polynomials and exponentials, but too lengthy to reproduce here). Steady forced quasiharmonic oscillations are obtained for the special case when $\sigma \to \infty$.

J. A. Nohel (Atlanta, Ga.)

1709:

Massera, J. L. A criterion for the existence of almost periodic solutions of certain systems of almost periodic differential equations. Bol. Fac. Ingen. Agrimens. Montevideo **6** (1957/58), 345–349. Also published as Fac. Ingen. Montevideo. Publ. Inst. Mat. Estadist. **3** (1958), 99–103. (Spanish. English summary)

The author considers the "triangular" system of equations

$$\dot{x}_m + a_{m1}(t)x_1 + \cdots + a_{mm}(t)x_m = f_m(t) \quad (m = 1, \cdots, n)$$

with real almost periodic $a_{pq}(t)$, and he poses the problem, apparently novel, under what conditions on the $a_{pq}(t)$ for every set of real almost periodic $f_m(t)$ there will be at least one almost periodic set of solutions x_m. He finds a necessary and sufficient condition, namely, that the mean values of the diagonal coefficients shall be 0; and the almost periodic solutions are then even unique. A related result if the $f_m(t)$ are replaced by more general expressions $f_m(x_1, \cdots, x_n, t)$ which vary slowly with the $x_1, \cdots, x_n$.

S. Bochner (Princeton, N.J.)

1710:

Bylov, B. F. On stability from above of the greatest characteristic index of a system of linear differential equations with almost-periodic coefficients. Mat. Sb. (N.S.) **48 (90)** (1959), 117–128. (Russian)

The author considers the systems (1) $dx/dt=A(t)x$ and (2) $dx/dt=A(t)x+F(t, x)$, where x is an n-vector, $A(t)$ is an $n \times n$ matrix with entries $a_{ij}(t)$ almost periodic in t, the perturbation function $F(t, x)$ is continuous in t and x, $F(t, 0)=0$, and for some constant $\delta > 0$,

$$(*) \quad |F_i(t, \bar{x}) - F_i(t, x)| \leq \delta \sum_j |\dot{\bar{x}}_j - x_j| \quad (i = 1, \cdots, n).$$

Let Λ be the largest characteristic exponent associated with the solutions of (1). Λ has property (S)—i.e., Λ is stable from above with respect to arbitrary perturbation functions of the type considered above—if for every $\varepsilon > 0$ there is a $\delta > 0$ such for each perturbation function satisfying (*) the largest characteristic exponent associated with the solutions of (2) does not exceed $\Lambda+\varepsilon$. Let $X(t, \tau)$ denote any $n \times n$ matrix of independent solutions of (1) such that $X(\tau, \tau)=E$. Let the norm of a matrix be the greatest Euclidean norm of any row or column. It is proved that Λ has property (S) if and only if for every $\alpha > 0$ there is a constant $B(\alpha)$ independent of τ such that $\|x(t, \tau)\| \leq B(\alpha)\exp(\Lambda+\alpha)(t-\tau)$. It is also shown that Λ has property (S) if for every $\gamma > 0$ there exist positive numbers $\{L_i\} \to \infty$ such that in each interval of length L_i there is at least one τ_i for which $\|A(t+\tau_i) - A(t)\| \leq \exp(-\gamma L_i)$. *C. S. Coleman* (Claremont, Calif.)

1711:

Palamodov, V. P. Conditions at infinity for correct solvability of a certain class of equations of the form $p(id/dx)u=f$. Dokl. Akad. Nauk SSSR **129** (1959), 740–743. (Russian)

1712:

Legatos, Gerassimos G. Contribution à l'étude du champ de directions des équations différentielles. Bull. Soc. Math. Grèce **31**, 1–33 (1959). (Greek. French summary)

1713:

Todorov, Iv. Équations différentielles linéaires à conditions limites généraux. Bŭlgar Akad. Nauk Izv. Mat. Inst. **4**, no. 1, 131–142 (1959). (Bulgarian. Russian and French summaries)

L'auteur construit une fonction de Green pour l'équation

$$L(u) \equiv u^{(n)} + p_1(x)u^{(n-1)} + \cdots + p_n(x)u = f(x),$$

à conditions limites $A_i(u)=0$. $A_i(u)$ sont des fonctionnelles linéaires telles que le déterminant $D[A_i(y_k)] \neq 0$.

Il donne des conditions nécessaires et suffisantes pour l'existence d'une telle fonction. *B. S. Popov* (Skopje)

1714:

Opial, Z. Sur les solutions de class (L^2) de l'équation différentielle $u''+q(t)u=0$. Ann. Polon. Math. **7** (1960), 293–303.

Consider the equation (1) $u''+q(t)u=0$. Let $Q(t)=\max(|q(s)|, 1)$, $0 \leq s \leq t$. Theorem 1 then states that if $u(t)$ is a solution of (1) such that $\int_0^\infty u^2(t)dt < \infty$, then $\int_0^\infty u'^2(t)dt/Q(t) < \infty$ also. Theorem 2 states that the same result is true for the equation (1') $u''+(\lambda+q(t))u=0$, λ an arbitrary constant, if it is assumed that $\left|\int_0^t q(t)dt\right| \leq \sqrt{Q(t)}$, $Q(t) \geq c > 0$, where now Q is continuous and nondecreasing for $t \geq 0$. The author gives a discussion of other results related to theorem 1, and mentions that theorem 2 is an improvement of a result due to Hartman [Amer. J. Math. **73** (1951), 635–645; MR **13**, 462]. The significance of this type of result is discussed by Wintner [ibid. **69** (1947), 5–13; MR **8**, 463] and Hartman and Wintner [ibid. **70** (1948), 295–308; MR **10**, 120].

A. Stokes (Baltimore, Md.)

1715:

Conti, Roberto. Equazioni differenziali ordinarie con condizioni lineari generali. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **26** (1959), 636–640.

Let E_n be the space of n-vector functions $x(t)$, $t \in \Delta = [t_0, t_0+h]$, R_n the subspace of constant n-vectors, and consider the n-vector equation (1) $x' = A(t)x + a(t, x)$ with the boundary condition (2) $\Phi x = c$, where Φ is a linear, bounded functional from E_n to R_n. Let A_n be the subspace of E_n consisting of the solutions of $x' = A(t)x$ and let Φ_1 be the restriction of Φ to A_n. Theorem: Suppose $A(t)$ is measurable in Δ, $\|A(t)\| \leq \mu(t)$, $\mu(t)$ integrable in Δ; Φ_1^{-1} exists; and in some region $\|x - \Phi_1^{-1}c\| \leq r$, $t \in \Delta$, the function $a(t, x)$ is continuous in x for every t and measurable in t for every x and $\|a(t, x)\| \leq \nu(t)$, $\nu(t)$ integrable in Δ. If

$$\exp\left(\int_\Delta \mu(t)dt\right) \int_\Delta \nu(t)dt \leq \frac{r}{1+\|\Phi_1^{-1}\Phi\|},$$

then there is at least one solution of (1) satisfying (2). The proof of this theorem consists in showing that a convenient transformation has a fixed point. For $A(t) \equiv 0$, the theorem reduces to a result of L. N. Ešukov [Uspehi Mat. Nauk **13** (1958), no. 3 (81), 191–196; MR **20** #3313]. The relationship of this theorem to a previous result of the author [Ann. Mat. Pura Appl. (4) **46** (1958), 109–130; MR **21** #4265] is also given.

J. K. Hale (Baltimore, Md.)

1716:

Thomas, Johannes. Über eine Verallgemeinerung der Liouvilleschen Transformation. Math. Nachr. **17**, 273–280 (1959).

The paper is concerned with the real differential equation

$$y'' + f(x, \lambda)y' + [\lambda g(x, \lambda) + h(x, \lambda)]y = 0,$$

where λ is a parameter. In addition to certain mild regularity conditions in a region $a \leq x \leq b$, $\lambda_1 \leq \lambda \leq \lambda_2$, the following two hypotheses are essential. (1) $g = (x-a)^\nu \times g_1(x, \lambda)$, $\nu \geq 0$, $g_1 > 0$, (2) $\mu(\lambda) = \pi^{-1-\frac{1}{2}\nu}\lambda \int_a^b \sqrt{(g(x, \lambda))}dx$ is strictly monotonic. The paper exhibits explicitly a continuous one-to-one transformation of x, y, λ into z, u, μ that reduces the differential equation to a simpler form more suitable for the study of the distribution of eigenvalues for boundary value problems. The simpler differential equation has the form

$$\frac{d^2u}{dz^2} + \left[\mu\left(\frac{\nu}{2}+1\right)^2 z^\nu + q(z, \mu)\right]u = 0.$$

For $\nu = 0$ the transformation reduces to one already used by Liouville. For positive ν, but for a slightly more special type of differential equation, a similar transformation was given by R. E. Langer [Trans. Amer. Math. Soc. **33** (1931), 23–64]. *W. Wasow* (Madison, Wis.)

1717:

Thomas, Johannes. Über Sturmsche Differentialgleichungen mit einer Anfangs- und einer integralen Nebenbedingung. Math. Nachr. **17**, 281–307 (1959).

In an earlier paper [see preceding review] the author showed that a fairly general class of differential equations can be transformed into the form

$$y''(x, \lambda) + [\lambda^2(\tfrac{1}{2}\nu+1)^2 x^\nu + h(x, \lambda)]y(x, \lambda) = 0$$

($\nu \geq 0$, $0 \leq x \leq \pi$, $\lambda > 0$). The present article is devoted to theorems concerning the location of eigenvalues λ for the problem consisting of this differential equation and the subsidiary conditions

$$\alpha(\lambda)y(0, \lambda) + \beta(\lambda)y'(0, \lambda) = 0,$$

$$\int_0^\pi [A(x, \lambda)y(x, \lambda) + B(x, \lambda)y'(x, \lambda)]dx = 0.$$

Here $\alpha(\lambda)$, $\beta(\lambda)$ are pointwise continuous; $h(x, \lambda)$ is continuous in x and continuous in the mean square sense in λ (i.e., $\lim_{\bar{\lambda}\to\lambda} \int_0^\pi [h(x, \bar{\lambda}) - h(x, \lambda)]^2 dx = 0$); $A(x, \lambda)$, $B(x, \lambda)$ are bounded and measurable in x and continuous in the mean square sense in λ.

The author derives conditions under which a given interval contains at least one eigenvalue. These conditions are too long to be stated here. To derive them the differential equation is first converted into a Volterra integral equation with the help of a fundamental system for the explicitly solvable special equation obtained when $h \equiv 0$. The subsidiary conditions then lead to a characteristic equation for the eigenvalues. The conditions imposed by the author guarantee the existence of a root of this

equation in the indicated interval on the basis of a continuity argument. When $\nu=0$ the kernel of the integral equation is a trigonometric function, while for $\nu>0$ certain cylinder functions are involved. In the latter case the simplifying assumption $B(x, \lambda)=0$ is added.

W. Wasow (Madison, Wis.)

1718:

Thomas, Johannes. Über Sturmsche Differentialgleichungen mit zwei integralen Nebenbedingungen. Math. Nachr. **20** (1959), 39–52.

The problem, the method and the results of this paper are analogous to the preceding one. The difference is that now the subsidiary conditions are

$$\int_0^\pi [A_i(x, \lambda)y(x, \lambda)+B_i(x, \lambda)y'(x, \lambda)]dx = 0 \quad (i = 1, 2).$$

If $\nu>0$ in the differential equation, only cases where $B_1(x, \lambda)=B_2(x, \lambda)\equiv 0$ are treated. The author's sufficient conditions for the existence of eigenvalues in a given interval are somewhat simpler than in the former paper but still too long to be quoted. In an appendix the result of the preceding paper is slightly improved for the case $\nu>0$.

W. Wasow (Madison, Wis.)

1719:

Bojadžiev, G. Periodische Bewegungen eines Systems von n Freiheitsgraden bei vielfachen Wurzeln der charakteristischen Gleichung. C. R. Acad. Bulgare Sci. **12** (1959), 5–8. (Russian summary)

This paper generalizes a result of J. Horn [Z. Math. Physik **48** (1903), 400–434] on the existence of periodic solutions of Euler-Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_\nu}\right)-\frac{\partial T}{\partial q_\nu}=\frac{\partial u}{\partial q_\nu} \quad (\nu = 1, 2, \cdots, n)$$

near the stable equilibrium solution $q_\nu=0$. The functions T, U are assumed to be real analytic and of the form

$$T=\tfrac{1}{2}\sum_{\nu,\mu=1}^{n} A_{\nu\mu}(q)\dot{q}_\nu\dot{q}_\mu, \quad U=\tfrac{1}{2}\sum_{\nu,\mu=1}^{n} B_{\nu\mu}(q)q_\nu q_\mu$$

with positive definite $A_{\nu\mu}(0)$, $B_{\nu\mu}(0)$. The author states the existence of periodic solutions even for the case that the characteristic roots of the linearized equations have integer quotients, at least under some additional conditions (formula (11)). Proofs are indicated, but not complete.

J. Moser (Cambridge, Mass.)

1720:

Opial, Z. Démonstration d'un théorème de N. Levinson et C. Langenhop. Ann. Polon. Math. **7** (1960), 241–246.

The author proves the existence of a periodic solution of the equation

$$x''+f(x, x')x'+g(x) = e(t) \quad (x' = dx/dt)$$

under the hypotheses: (1) $f(x, v)\geq m>0$ for $|x|\geq a$, $|v|\geq a$ $(a>0)$; (2) $f(x, v)\geq -M$ for all real x and v $(M>0)$; (3) $e(t+\omega)=e(t)$ $(\omega>0)$; and (4) $\liminf g(x)>Ma+E$ as $x\to+\infty$, $\limsup g(x)<-(Ma+E)$ as $x\to-\infty$, where $E=\max|e(t)|$. This generalizes the theorem of N. Levinson [J. Math. and Phys. **22** (1943), 41–48; MR **5**, 66] as strengthened by the reviewer [ibid. **30** (1951), 36–39; MR **13**, 238], and the proof is a uniform treatment of these two cases. The author also gives an example to show that the constant $Ma+E$ in condition (4) can not be improved.

C. E. Langenhop (Princeton, N.J.)

1721:

Turán, P. On a property of the stable or conditionally stable solutions of systems of non-linear differential equations. Ann. Mat. Pura Appl. (4) **48** (1959), 333–340.

Consider the vector differential equation (1) $\dot{x}=Ax+W(x, t)$, where A is an $n\times n$ constant matrix, such that all the eigenvalues of A have negative real parts. Under suitable conditions on W (e.g., $\|W(x, t)\|=o(\|x\|)$ as $\|x\|\to 0$, uniformly in t), it is well known that every solution $x(t)$ of (1), with $\|x(0)\|$ sufficiently small, tends to zero as $t\to\infty$. Further, there exists a $\sigma>0$ such that for any such solution $x(t)$, $\limsup(\log\|x(t)\|/t)\geq -\sigma$, as $t\to\infty$. Thus, for any $\varepsilon>0$, there exists a sequence $\{t_k\}$, with $t_k\to\infty$ as $k\to\infty$, so that (2) $\|x(t_k)\|\geq\exp\{-(\sigma+\varepsilon)t_k\}$, for every k. Under the above conditions on A and W, the author shows that any such sequence $\{t_k\}$ cannot have arbitrarily large gaps. More precisely, he demonstrates the existence of a number $a>0$ such that the inequality (2) is satisfied for at least one value of t in every interval of length a, say $[t_0, t_0+a]$, for t_0 sufficiently large. The author actually proves the theorem assuming only that $\|W(x, t)\|\leq\eta\|x\|$, for x in a certain neighborhood of the origin, where η and a are given explicitly in terms of the eigenvalues of A and the choice of ε. {There are several minor misprints, among them the omission of the word "almost" before "vanishes" in line 15 from the bottom of page 334.}

A. Stokes (Baltimore, Md.)

1722:

Lillo, James C. Perturbations of nonlinear systems. Acta Math. **103** (1960), 123–138.

The author secures conditions on $A(t)$ sufficient to guarantee that the characteristic exponents of the system $x'=[A(t)+B(t)]x$, where $A(t)$ and $B(t)$ are continuous and uniformly bounded matrices, are continuous at $B(t)\equiv 0$ as functions of $B(t)$. The paper also contains results concerning the existence and variations of bounded solutions of certain nonlinear equations and their application to almost periodic nonlinear systems. This paper is an extension of an earlier paper by the same author [Ann. of Math. (2) **69** (1959), 467–485; MR **21** #177].

W. R. Utz (Columbia, Mo.)

1723:

Iwano, Masahiro. Intégration analytique d'un système d'équations différentielles non linéaires dans le voisinage d'un point singulier. II. Ann. Mat. Pura Appl. (4) **47** (1959), 91–149.

This is a direct continuation of a memoir by the same author, which has been reviewed previously [Ann. Mat. Pura Appl. (4) **44** (1957), 261–292; MR **20** #3320]. The same system of differential equations is under consideration; and the author continues to seek conditions under which certain formal infinite series converge, and yield actual solutions of the differential equations. As in the case of the first memoir, the analysis and results are so complicated that no adequate description of them is possible here.

L. A. MacColl (New York)

1724:

Phakadze, A. V.; Šestakov, A. A. On the classification of the singular points of a first order differential equation not solved for the derivative. Mat. Sb. (N.S.) **49 (91)** (1959), 3–12. (Russian)

The authors consider the equation (1) $F(x, y, p)=0$, where $p=y'$ and F has continuous partial derivatives of the first three orders. A solution of the system, $F=0$, $F_p'=0$, $F_x'+pF_y'=0$, is called a singular point of (1). Apparently, this differs from the definition given by Petrovskiĭ [*Lekcii po teorii obyknovennyh differencial'nyh uravnenii*, 3d ed., GITTL, Moscow-Leningrad, 1949; MR **12**, 334]. It is assumed that the singular points are isolated in the x, y, p-space. The authors first classify the singular points of the equation of first approximation —i.e., the equation using up through the 2nd order terms of the Taylor series of F—and then apply this classification to the singular points of (1). It is claimed that the qualitative behavior of the integral curves near the singular points is the same for both equations. The authors state that their analysis clarifies the Poincaré classification of singular points of an equation solved for p. {It should be noted that the Poincaré singular points have quite different properties from those analyzed in this paper. There are a number of notational errors.}

C. S. Coleman (Claremont, Calif.)

1725:

Ważewski, T. Sur un problème asymptotique relatif au système de deux équations différentielles ordinaires. Ann. Mat. Pura Appl. (4) **49** (1960), 139–146.

Hypothesis H_1: Suppose D is an open connected bounded set in the plane whose boundary F consists of $p+1$ analytic, disjoint closed Jordan curves $J_0, J_1, \cdots, J_p$, with D in the interior of J_0 and in the exterior of $J_1, \cdots, J_p$, and that $D+F\subset\Omega$ where Ω is open. Hypothesis H_2: The two dimensional system (1) $dx/dt=R(x)$ is such that $R(x)$ is continuous in Ω, and for each set of initial values $t=t^*$, $x=x^*$, there is a unique integral curve $x=\gamma(t, t^*, x^*)$ and if $x^*\in F$ then x^* is a point of strict egress. Let D satisfy H_1. A set W is said to have property Prop(H_1, D, W) if $W\subset D$ and W is a continuum which decomposes the plane into exactly $p+1$ open connected disjoint sets, $D_0, \cdots, D_p$ with $J_i\subset D_i$ $(i=0, 1, \cdots, p)$. The main theorem of the paper is the following. Suppose H_1, H_2 are satisfied. If W is the asymptotic set of D (integral curves of (1) which remain in D), then W satisfies Prop(H_1, D, W). Conversely, if H_1 is satisfied and W satisfies Prop(H_1, D, W), then there exists a differential system (1) satisfying H_2 and W is the asymptotic set. The theorem is proved by using the idea of an asymptotic retract [see A. Pliś, Bull. Acad. Polon. Sci. Cl. III **2** (1954), 415–418; MR **16**, 700] and then showing that asymptotic retract and Prop(H_1, D, W) are equivalent. The author states the difficulties that are associated with an extension of this theorem to systems of dimension $n>2$.

J. K. Hale (Baltimore, Md.)

1726:

Aumann, G. Beispiele zur Stabilität und Instabilität von Bewegungen. Math. Z. **73** (1960), 374–375.

The systems of equations (I) $\dot{x}=-x(1-y)$, $\dot{y}=y(1-x)$ and (II) $\dot{x}=x(1-y)$, $\dot{y}=y(1-x)$, are discussed as examples of differential equations with stable and unstable equilibrium respectively, and interpreted as equations for populations (I) with x predatory and living on y alone, (II) with each predatory on the other but possessing other resources.

J. L. B. Cooper (Cardiff)

1727:

Ura, Taro. Sur le courant extérieur à une région invariante. Prolongements d'une caractéristique et l'ordre de stabilité. Funkcial. Ekvac. **2** (1959), 143–200.

In an earlier paper [Ann. Sci. École Norm. Sup. (3) **70** (1953), 287–360; MR **16**, 247] the author considered dynamical systems given by a system of differential equations valid in a bounded domain of a euclidean space. The concepts of stability of trajectories with respect to the closure of an invariant region given there are here extended to certain one-parameter transformation groups. According to the author, the paper consists of a French translation, with corrections, of papers previously published in Japanese [Sûgaku **9** (1957/58), 137–148, 218–235; MR **21** #7334, #7335].

W. R. Utz (Columbia, Mo.)

1728:

Veksler, D. [Wexler, D.] Stability theorems for a system of stationary differential equations. Rev. Math. Pures Appl. **3** (1958), 131–138. (Russian)

Consider an n-vector system (1) $\dot{x}=X(x)$, $X(0)=0$, where X is defined, continuous and Lipschitz in a domain D of E^n which contains the origin. Stability of the origin is referred, in this paper, to Liapunov-like functions $V(x)$ satisfying broader conditions than those that are standard by now.

Denote by S_h the spherical region of center O $(x=0)$ and radius h. Let $x_0\in D$ and let $f(x_0, t)$ denote the path through x_0 for $t\geq 0$. If $P\subset D$ then let $f(P, t)$ denote the set of all paths issued from P. If P contains all $f(P, t)$ it is said to be invariant. The origin O is stable for (1) relative to the invariant set P whenever: (a) $O\in P$; (b) given $0<\varepsilon$ there is an $\eta(\varepsilon)$ such that if $x_0\in S_\eta\cap P$ then $f(x_0, t)\in S_\varepsilon$, $0\leq t<+\infty$. If in addition $f(x_0; t)\to O$ as $t\to+\infty$, the stability is asymptotic. The latter is uniform as to the initial point x_0 if $f(x_0, t)\to O$ uniformly as to x_0 when $t\to+\infty$. Theorem 1: Given $S_h\subset D$ and N a closed invariant set relative to which the origin is asymptotically stable. Let $V(x)$, $V(0)=0$, be continuous and $\neq 0$ in S_h-N. Take any $x_0\in S_h-N$ and $0<t_1<t_2$ such that $f(x_0, t_1)$, $f(x_0, t_2)$ are in the set where V is defined. Set $V_i=V(f(x_0, t_i))$. Then if $V_1>0$ implies $V_2\leq V_1$, or else $V_1<0$ implies $V_2\geq V_1$, the origin is stable. Theorem 2: Under the same conditions as in theorem 1, if no manifold $V(x)=C\neq 0$ contains a semi-path the origin is asymptotically stable. Theorem 3: Let the conditions of theorem 2 hold throughout E^n. Suppose also that if E^n-N is unbounded then V is infinitely great in that set. Then the origin is stable in the large.

Let now $n=3$, and let the components X_i of X be of class C^1 throughout E^3 and homogeneous of degree m. Then

$$V(x)=\sum(x_iX-x_kX_i)^2$$

is a suitable function V guaranteeing stability in the large.

S. Lefschetz (Mexico City)

1729:

Moriguchi, Haruo. An improvement of the WKB method in the presence of turning points and the asymptotic solutions of a class of Hill equations. J. Phys. Soc. Japan **14** (1959), 1771–1796.

In the differential equation

$$d^2y/dx^2-(\lambda+h^2f(x))y = 0,$$

let h be a large parameter and λ a constant or a function of h such that $\lambda=o(h^2)$, as $h\to\infty$. The author's aim is to derive asymptotic expressions for the solutions, as $h\to\infty$, that are valid in regions containing zeros of the coefficient of y. This is done by a sequence of transformation of the variables. The first transformation is of a type frequently employed to reduce turning point problems to differential equations that differ only slightly from some equations whose asymptotic theory is already known. The second transformation is a suitable small translation of the independent variable, depending on h, which restores some of the features of the original equation. Then the procedure is repeated with the equation so obtained. The translation is the essentially new feature of the method. It enables the author to improve existing results for second order turning points and to derive asymptotic expressions for the solutions of certain Hill equations in terms of Mathieu functions.

The question of whether the expressions obtained by formal arguments are actually uniform asymptotic approximations is not discussed except for the remark that it is easy. *W. Wasow* (Madison, Wis.)

1730:

Vasil'eva, A. B. Asymptotic formulas for solving ordinary simultaneous differential equations having derivatives with parameters of different order of smallness. Dokl. Akad. Nauk SSSR **128** (1959), 1110–1113. (Russian)

The author considers systems of the form (*) $\mu_1\mu_2\,dz/dt=w(z,y,x,t)$, $\mu_2\,dy/dt=v(z,y,x,t)$, $dx/dt=u(z,y,x,t)$, $z|_{t=0}=z^0$, $y|_{t=0}=y^0$, $x|_{t=0}=x^0$, where μ_1 and μ_2 are $\geqq 0$ and z, y and x may be finite dimensional vectors. Using a theorem of Tihonov [Mat. Sb. (N.S.) **31 (73)** (1952), 575–586; MR **14**, 1085] and some systems related to (*), the author indicates how to obtain asymptotic formulas Z_n, Y_n, X_n for a certain solution of (*) as μ_1 and $\mu_2\to 0$. If the actual solution of (*) is given by the functions z, y, x, then when $0\leq t\leq T$, $0\leq\mu_i\leq\mu_i^0$, $i=1,2$ one has $|z-Z_n|\leq C\sum_{k=0}^{n+1}\mu_1^{n+1-k}\mu_2^k$ and similar bounds for $y-Y_n$ and $x-X_n$, where C is independent of t, μ_1 and μ_2. The result is a generalization of an earlier result by the author [same Dokl. **124** (1959), 509–512; MR **21** #7341].

C. E. Langenhop (Princeton, N.J.)

1731:

Saito, Tomoyuki; Oshida, Isao. The W. K. B. method for the differential equations of the fourth order. J. Phys. Soc. Japan **14** (1959), 1816–1819.

Approximate expressions are derived for the solutions of the differential equation

$$a(x)y'''+f(x)y''-y''+g(x)y'-b(x)y = 0$$

under various assumptions on the relative smallness of the coefficients. Only formal arguments are given.

W. Wasow (Madison, Wis.)

1732:

Goering, Herbert. Mathematische Untersuchungen zum Umschlag laminar-turbulent. Math. Nachr. **17**, 358–393 (1959).

The Orr-Sommerfeld equation

$$\phi^{(4)}(y)-i\alpha R\{(U(y)-c)\phi''(y) -[\alpha^2(U(y)-c)+U''(y)]\phi(y)\} = 0$$

for small periodic disturbances in a viscous laminar flow is studied asymptotically for large values of the Reynolds number R. The given velocity profile $U(y)$ is assumed to have the form characteristic for boundary layer flow: It is zero at the wall and constant for large y. The parameters α and c are assumed positive and independent of R, so that the stream function $x=\phi(y)\exp[i\alpha(x-ct)]$ is periodic in time and space. The procedure is modelled after the investigations of Tollmien [Z. Angew. Math. Mech. **25/27** (1947), 33–50, 70–83; MR **9**, 476]. First two nonlinear solutions $\phi_3(y)$, $\phi_4(y)$ of the simplified equation $\phi^{(4)}-i\alpha R(U-c)\phi''=0$ are calculated asymptotically, for large R, with the help of the asymptotic formulas for Hankel functions and the variation of parameters method. Then it is shown that the full differential equation possesses two solutions ϕ_{III}, ϕ_{IV} which are asymptotically represented by ϕ_3, ϕ_4, uniformly through the whole cross section of the flow. After this, reference is made to a result of this reviewer [Ann. of Math. (2) **49** (1948), 852–871; MR **10**, 377] which states that there exist two more solutions ϕ_I, ϕ_{II} which are asymptotically represented by solutions ϕ_1, ϕ_2 of the inviscid equation $(U-c)\phi''-[\alpha^2(U-c)+U'']\phi=0$. These representations are known to be valid in a region of the y-axis obtained by deleting a neighborhood of the point where $U(y)=c$ from a sufficiently small interval I containing this point. The author needs this result in an interval that may be larger than I. In order to prove this extension he replaces $U(y)$ temporarily by a polynomial and represents the solution of the full equation by Laplace contour integrals which are then studied asymptotically. The asymptotic nature of ϕ_I, ϕ_{II} in the neighborhood of the point where $U(y)=c$ is not discussed. *W. Wasow* (Madison, Wis.)

1733:

Lin, C. C.; Rabenstein, A. L. On the asymptotic solutions of a class of ordinary differential equations of the fourth order. I. Existence of regular formal solutions. Trans. Amer. Math. Soc. **94** (1960), 24–57.

The equations studied are those of the form

$$\mathscr{L}\phi = \sum_{j=0}^{4}\lambda^2P_j(x,\lambda)d^j\phi/dx^j.$$

The coefficients P_j are assumed to be analytic in x about the origin and analytic in λ at ∞ (below, such functions are referred to as functions of class $\mathscr{A}$) with $P_4=\lambda^{-2}$ and $P_3=0$. The matter at issue is the behavior of solutions of $\mathscr{L}\phi=0$ as $|\lambda|\to\infty$ under the hypothesis that $P_2(x,\infty)$ has a simple zero at the origin. Langer [same Trans. **84** (1957), 144–191; MR **18**, 738], has previously given a description of this behavior under the general hypotheses assumed by the authors, and Wasow [Ann. of Math. (2) **58** (1953), 222–252; MR **15**, 874], has studied the behavior of the solutions of the differential equation of hydrodynamic stability, which is a special case of $\mathscr{L}\phi=0$. Langer's theory is based on the third order equation $v'''+\lambda^2(xv'+\alpha v)=0$ in which α

is analytic in λ at ∞, while the authors' theory is based directly on the fourth order equation $u^{iv}+\lambda^2(zu''+\alpha(\lambda)u'+\beta(\lambda)u)=0$, which is more obviously related to $\mathscr{L}\phi=0$. In the event that α is an integer or zero a solution of $zu''+\alpha u'+\beta u=0$ may have a logarithmic singularity. It is necessary to allow this case if one is to obtain meaningful information relative to stability of solutions of the Orr-Sommerfeld equation. Wasow considered only the case $\alpha=0$, while the condition that $\alpha(\infty)$ be an integer leads to an exceptional class of equations to which Langer's theory is inapplicable, and without further investigation it and the present theory cannot be compared in this circumstance. Otherwise, they should lead to equivalent results.

While the exposition of the existence theory of formal asymptotic solutions to $\mathscr{L}\phi=0$ is lengthy and intricate, the authors are always careful to refresh the reader with motivation for their delicate choices and manipulations. In rough outline, this theory is constructed as follows. New independent variables

$$z=[(3/2)\int_0^z P_2^{1/2}(x,\infty)dx]^{2/3},\quad \psi=\varphi[P_2(x,\infty)/z]^{3/4}$$

are introduced in $\mathscr{L}\phi=0$ to the end that the leading coefficient of ψ'' in the resulting equation $\psi^{iv}+\lambda^2(p\psi''+q\psi'+r\psi)=0$ for ψ be z. Next, a formal solution χ of the form $A\psi+B\psi'+\lambda^{-1}C\psi''+\lambda^{-2}D\psi'''$ is postulated in which the coefficients A, B, C, and D are to lie in class $\mathscr{A}$. The first determining condition on these coefficients is that the inverse of the transformation from ψ to χ be analytic. They are then further determined so that χ satisfy an equation of the form

$$\chi^{iv}+\lambda^2(z\chi''+\alpha(\infty)\chi'+\beta(\infty)\chi)=\lambda(a\chi+b\chi'+\lambda^{-1}c\chi''+\lambda^{-2}d\chi'''),$$

where a, b, c, and d are also in class $\mathscr{A}$. The construction of asymptotic solutions of this "normalized" equation is then carried out. The last step is to write a solution ψ as a linear combination of χ and its derivatives with analytic coefficients.

It is not possible to execute this program via a constructive algorithm which reduces the task of finding the coefficients A, B, C, and D to successive quadratures as in Langer's theory. In the present one the burden is ultimately thrown on the solution of the two second order equations

$$zf''+\alpha(\infty)f'+\beta(\infty)f=0,\quad zf''+q(z,\infty)f'+r(z,\infty)f=0,$$

for which integrals are, in general, unknown, so that power series solutions of these equations are used. The particular features of the theory which delineate its dependence on p and q and which thus bring out the properties of solutions important in the hydrodynamical case are engendered by the possible singular nature of solutions of these reduced equations.

The questions of existence of actual solutions approximated by truncations of the formal solutions whose existence is demonstrated in this paper, and of applications to the hydrodynamical problem, are to be dealt with in two subsequent papers. *N. D. Kazarinoff* (Moscow)

1734:

Zadiraka, K. V. On a periodic integral manifold of a nonlinear system of differential equations with a small parameter in the derivatives. Ukrain. Mat. Ž. **11** (1959), 243–250. (Russian. English summary)

This paper continues the author's study of the system $x'=f(t,x,z)$; $\mu z'=F(t,x,z)$. In a previous paper [Dopovidi Akad. Nauk Ukrain. RSR **1958**, 131–134, MR **20** #5331] he studied existence and uniqueness of periodic solutions related to a solution for $\mu=0$. Here he studies the regularity properties of these solutions.

R. R. D. Kemp (Kingston, Ont.)

PARTIAL DIFFERENTIAL EQUATIONS

See also 1770.

1735:

Douglas, Jim, Jr. An unusual stability criterion for a difference analogue of the heat equation. J. Math. Phys. **38** (1959/60), 150–152.

The author exhibits a difference equation approximation to the heat equation in two space variables for which the following relation between the space increment h and the time increment k is necessary for stability and convergence: $5/4\leq kh^{-2}<\infty$. *J. Elliott* (New York)

1736:

Gel'fand, I. M. Some problems in the theory of quasilinear equations. Uspehi Mat. Nauk **14** (1959), no. 2 (86), 87–158. (Russian)

The content of this paper is that of a lecture course given by the author at Moscow State University in 1957–1958. Pointing out that a general theory of quasilinear equations does not exist, he has provided a survey of certain results in the field without attempting a rigorous development. For the latter one should consult O. A. Oleĭnik [same Uspehi **12** (1957), no. 3 (75), 3–73; MR **20** #1055].

Systems of differential equations containing time as an independent variable and for which a Cauchy problem is formulated and correctly set are called evolutionary. When the coefficients are functions of the dependent variables, solutions may exhibit discontinuities even when the initial data are arbitrarily smooth. The author's canonical case for quasilinear evolutionary systems is the system (*) $\partial u/\partial t+A(u)(\partial u/\partial x)=0$, in which $A(u)du=df(u)$, and he imposes the requirement that if one replaces the right member by a "viscosity" term $\varepsilon B(\partial^2 u/\partial x^2)$, then the discontinuous solutions must be admissible, i.e., obtainable as limits of continuous solutions of the new system as $\varepsilon\to 0$. A necessary condition is shown to be $-\omega(u^+-u^-)+f(u^+)-f(u^-)=0$, where $u=u^+$ for $x>\omega t$ and $u=u^-$ for $x<\omega t$.

After a digression to develop the equations of motion of a viscous fluid, the author returns to consider sufficient conditions for admissibility, first for a single equation and then for systems. He treats the problem of constructing an admissible solution of (*) satisfying the discontinuous initial data $u(x,0)=u^-$ for $x<0$, u^+ for $x>0$, where u^- and u^+ are arbitrary constants. For a system consisting of a single equation, the problem always has a unique solution; otherwise the system must have convexity properties. After this he considers the analogous problems for evolutionary linear systems in two independent variables.

In a final portion he passes to specific problems from magnetohydrodynamics, spontaneous combustion, chemical reactions, and the propagation of flames.

R. N. Goss (San Diego, Calif.)

1737:

Ghelfand, I. M. [Gel'fand, I. M.] Some problems of the theory of quasilinear equations. Acad. R. P. Romîne. An. Romîno-Soviet. Ser. Mat.-Fiz. (3) **13** (1959), no. 4 (31), 5–83. (Romanian)

Translated from the paper reviewed above.

1738:

Hornich, Hans. Zum Existenzproblem der linearen partiellen Differentialgleichungen. Monatsch. Math. **63** (1959), 378–393.

Continuing his investigation of the linear first order partial differential equation in 2 independent variables, the author establishes the following existence theorem. Let $\alpha(x, y)$ be a uniformly continuous function in a bounded, simply-connected region G; for the solutions of the ordinary differential equation (2) $dx:dy = \cos\alpha(x, y):\sin\alpha(x, y)$, let the following two conditions hold. (A) There is no closed solution-path of (2) on G and on its closed hull $\bar{G}$. (B) If the solution component K consists of the denumerably many solution complexes L_n, and σ_n is the supremum of the lengths of all solution-paths between any 2 points of L_n, then $\sum \sigma_n$ converges and is uniformly bounded for all K. Then there exists a solution in G of the equation

$$(1) \qquad \cos\alpha(x, y)\cdot U_x + \sin\alpha(x, y)\cdot U_y = f(x, y)$$

for every uniformly continuous function f in G. Here a solution does not mean a function of class C^1 (the well-known example of Perron disproves this) but a function U continuous in G which at every point of G has a directional derivative in the direction making an angle $\alpha(x, y)$ with the x-axis, which is equal to $f(x, y)$. A solution-path of (2) is a continuous rectifiable curve: $x = x(s)$, $y = y(s)$, where the functions $x(s)$ and $y(s)$, which are differentiable almost everywhere, satisfy (2) for almost all values of s; a solution complex belonging to a point p of G is the largest subset $L(p)$ of G so that one can join every point of $L(p)$ to p by a solution-path of (2). By means of three interesting examples, he shows that if G is not simply connected or if (B) is not true, one can find an f for which (1) is not solvable. He proves, under the hypotheses of the existence theorem, that for each set of solution components dense in G one can arbitrarily assign the values of U at a certain point of each of these solution components. Finally, for the homogeneous equation, i.e., when $f(x, y) = 0$, there is a non-trivial solution if and only if more than one solution-component exists in G. *D. L. Bernstein* (Towson, Md.)

1739:

Volpato, Mario. Sul problema di Cauchy per equazioni differenziali quasi lineari alle derivate parziali del primo ordine. Rend. Sem. Mat. Univ. Padova **28** (1958), 244–262.

L'A. studia il problema di Cauchy per l'equazione quasi lineare

$$(1) \quad \frac{\partial z}{\partial x} + \sum_1^n {}_r f_r(x, y_1, \cdots, y_n, z, \lambda)\frac{\partial z}{\partial y_r} = f_{n+1}(x, y_1, \cdots, y_n, z, \lambda)$$

con le condizioni iniziali (per $x = \xi$):

$$z(\xi, y_1, \cdots, y_r, \xi, \lambda) = \phi(y_1, \cdots, y_n, \lambda) \qquad (-\infty < y_1, \cdots, y_n, \lambda < +\infty)$$

nella classe A delle funzioni $z(x, y_1, \cdots, y_n, \xi, \lambda)$ lipschitziane rispetto a $y_1, \cdots, y_n$, λ, continue in $S_*'(a \leqq x, \xi \leqq b, -\infty < y_1, \cdots, y_n, \lambda < +\infty)$ assolutamente continue rispetto a x e a ξ, con derivate z_*', z_t' sommabili in S_*'. La (1) si intende soddisfatta quasi ovunque in S_*'.

L'esistenza di una soluzione si riduce alla ricerca di un elemento unito per una trasformazione funzionale, ed è provata nell'ipotesi che le f_i e ϕ soddisfino alle condizioni di Lipschitz:

$$|f_i(x, u_1, \cdots, u_n, z, \lambda) - f_i(x, U_1, \cdots, U_n, Z, \Lambda)| \leq P_i(x)\{\sum_1^n {}_r |u_r - U_r| + |z - Z| + |\lambda - \Lambda|\}$$

$$(i = 1, \cdots, n+i; \int_a^b P_i(x)dx < +\infty);$$

$$|\phi(u_1, \cdots, u_n, \lambda) - \phi(U_1, \cdots, U_n, \Lambda)| \leq h\{\sum_1^n {}_r |u_r - U_r| + |\lambda - \Lambda|\}$$

$$(3(h+1)\int_a^b \sum_1^{n+1} {}_i P_i(x)dx < 1).$$

Un secondo enunciato concerne il sistema caratteristico, che risulta soddisfatto. *L. Amerio* (Milan)

1740:

Kuznecov, N. N. Some asymptotic properties of the generalized solution of the Cauchy problem for a quasilinear equation of first order. Uspehi Mat. Nauk **14** (1959), no. 2 (86), 203–209. (Russian)

A classification of equations of the type

$$(*) \qquad \frac{\partial v}{\partial t} + \frac{\partial A(v, t)}{\partial x} = \alpha(t)v + \psi(t, x)$$

is sought, which discriminates according to whether a generalized solution of the Cauchy problem has none, one, or more than one nonvanishing discontinuities as $t \to \infty$. An equation (*) is of type (1) if all smooth initial data solutions have no more than one nonvanishing discontinuity as $t \to \infty$. Such an equation is of type (2) if all discontinuities disappear as $t \to \infty$, and of type (3) if more than one discontinuity does not vanish as $t \to \infty$. It is assumed that A lies in C^2 of the space-time domain where $t \geqq 0$ and $-\infty < x < \infty$, that A_{vv}'' is bounded and positive definite and that α and ψ satisfy various smoothness and other conditions. Under these hypotheses, necessary and sufficient conditions are found which determine the type of (*). If (*) is of type (2) and if v and $\bar{v}$ are generalized solutions corresponding to bounded initial data v_0 and $\bar{v}_0$, then, given $\varepsilon > 0$, there can be found a $T(\varepsilon)$ such that if $t > T(\varepsilon)$ then $|v(x, t) - \bar{v}(x, t)| < \varepsilon$. A similar theorem is given for equations of type (1).

N. D. Kazarinoff (Moscow)

1741:

Albrecht, Rudolf. Über Systeme partieller Differentialgleichungen erster Ordnung mit Cauchyschen Charakteristiken. Bayer. Akad. Wiss. Math.-Nat. Kl. Abh. **94** (1959), 44 pp.

Il s'agit ici des systèmes de m équations aux dérivées partielles du premier ordre à m fonctions inconnues de $n+1$ variables indépendantes (champ réel) pour lesquels il existe des caractéristiques de Cauchy. L'auteur donne

(en utilisant les notions introduites par E. Cartan) les conditions pour qu'un élément intégral à p dimensions ($1 \leq p \leq n$) soit caractéristique. Les systèmes qu'il étudie ont, en tout point intégral du domaine donné, m éléments linéaires intégraux dont les coefficients de direction proviennent d'un problème de valeurs propres d'une paire de matrices pour lesquelles il y a m vecteurs propres linéairement indépendants. Se bornant ensuite au cas d'un système quasilinéaire, il donne, pour le système des $2n+m+1$ équations différentielles ordinaires définissant les lignes caractéristiques, un théorème d'existence et d'unicité faisant intervenir des conditions de Lipschitz. Ce système permet alors de résoudre le problème de Cauchy pour le système d'équations aux dérivées partielles quasilinéaire considéré plus haut. A la fin, un exemple fort simple montre que le système différentiel ordinaire auquel on est conduit peut effectivement donner la solution du problème de Cauchy pour certains systèmes n'appartenant pas à la classe considérée ici, mais cela seulement, alors, pour certains choix particuliers de conditions initiales.

M. Janet (Paris)

1742:

Langer, Rudolph E. (Editor). ★Boundary problems in differential equations. Proceedings of a symposium conducted by the Mathematics Research Center at the University of Wisconsin, Madison, April 20–22, 1959. University of Wisconsin Press, Madison, Wis., 1960. x+324 pp. $4.00.

This volume contains texts of 19 lectures delivered at the symposium. These will be reviewed individually in Mathematical Reviews.

1743:

Payne, L. E. Bounds in the Cauchy problem for the Laplace equation. Arch. Rational Mech. Anal. **5**, 35–45 (1960).

Let u be a solution of Laplace equation $\Delta u = 0$ in a bounded n-dimensional domain D and let Σ be a portion of D. If $|u| \leq M$ in D, then for any $P \in D$, $|u(P)|^2 \leq K_0 M^{2(1-\delta)}[K_1 \int_\Sigma u^2 d\sigma + K_2 \int_\Sigma |\operatorname{grad} u|^2 d\sigma]^\delta$, where the K_i are constants depending only on Σ, D, and δ is a function of Σ, D, P, $0 < \delta < 1$. The significance of this inequality is that it establishes the continuous dependence, in the uniform sense, of solutions of $\Delta u = 0$, on the Cauchy data of u on Σ, these data being considered in the $L^2(\Sigma)$ norm. Inequalities of a similar nature, but with $\int_\Sigma u^2 d\sigma$, $\int_\Sigma |\operatorname{grad} u|^2 d\sigma$ replaced by $\sup_\Sigma u^2$, $\sup_\Sigma |\operatorname{grad} u|^2$, have been derived by M. M. Lavrent'ev [Izv. Akad. Nauk SSSR. Ser. Mat. **20** (1956), 819–842; Dokl. Akad. Nauk SSSR **112** (1957), 195–197; MR **19**, 426, 553].

A. Friedman (Minneapolis, Minn.)

1744:

Miranda, Carlo. Su alcuni aspetti della teoria delle equazioni ellittiche. Bull. Soc. Math. France **86** (1958), 331–354.

Il lavoro riproduce una interessante conferenza tenuta dall'Autore alla Riunione dei matematici di espressione latina [Nice, 1957] e offre una sintesi ben riuscita dei principali contributi portati in Italia, nei quindici anni precedenti il 1957, alla teoria delle equazioni alle derivate parziali di tipo ellittico. Per poter dare anche un'idea dei procedimenti adottati nello studio dei problemi al contorno, viene preso in considerazione particolarmente il problema "misto" di Dirichlet-Neumann per l'equazione lineare del secondo ordine in un aperto Ω di frontiera Γ:

$$\sum_{i,k}^{1,m} a_{ik} \frac{\partial^2 u}{\partial x_i \partial x_k} + \sum_i^{1,m} b_i \frac{\partial u}{\partial x_i} + cu = f \quad \text{in } \Omega,$$

$u = \alpha$ su Γ_1, $du/d\nu = \beta$ su $\Gamma - \Gamma_1$, dove Γ_1 è una porzione di Γ e $du/d\nu$ indica la cosidetta derivata "conormale" o "trasversale"; e in modo rapido ma assai chiaro vengono esposti, con i relativi risultati, i procedimenti di G. Fichera [Ann. Scuola Norm. Sup. Pisa (3) **1** (1947), 75–100; *Conv. Int. Equaz. Lin. Derivate Parz.*, pp. 174–227, Edizioni Cremonese, Rome, 1955; MR **11**, 724; **17**, 626], G. Stampacchia [Ann. Mat. Pura Appl. (4) **40** (1955), 193–209; MR **19**, 750], G. Cimmino [Rend. Circ. Mat. Palermo **61** (1937), 177–221]; vedi anche E. Magenes [Ann. Scuola Norm. Sup. Pisa (3) **8** (1954), 93–120; **9** (1955), 161–200; MR **16**, 365; **18**, 45] e infine dell'Autore stesso [Ann. Mat. Pura Appl. (4) **39** (1955), 279–303; MR **17**, 1213].

La bibliografia finale è completa e si estende anche ai lavori relativi alle equazioni paraboliche di secondo ordine.

E. Magenes (Pavia)

1745:

Lions, J. L. On the generalized radiation problem of Weinstein. J. Math. Mech. **8** (1959), 873–888.

Le problème considéré consiste à chercher une solution de

$$u_{xx} - u_{yy} - ky^{-1}u_y = 0, \tag{1}$$

k complexe, définie dans $y > 0$, (2) nulle pour $x \leq y$ et (3) prenant des valeurs données pour $x > 0$, $y = 0$. L'auteur fournit une solution complète de ce problème et sous les hypothèses les plus générales par application d'un opérateur transformant une solution de (1) vérifiant (2) en une solution de l'équation des ondes. On est ainsi à même de former toutes les solutions de (1) vérifiant (2), et de prouver l'existence et l'unicité de la solution du problème envisagé si $\mathrm{R}\{k\} < 1$, et la non existence de cette solution pour $\mathrm{R}\{k\} \geq 1$. La transformation fondamentale utilisée est étroitement liée à une nouvelle classe d'opérateurs de transmutations étudiée au cours du présent travail.

P. Germain (Paris)

1746:

Alekseev, A. S. Problems of Lambert type for the wave equation in a linear-homogeneous half-space. Leningrad. Gos. Univ. Uč. Zap. Ser. Mat. Nauk **32** (1958), 167–227. (Russian)

1747:

Owens, O. G. A characterization in E^3 of the everywhere regular solution of the reduced wave equation. Duke Math. J. **27** (1960), 81–90.

It is proved that there exists an everywhere defined unique C^2 solution on E^3 of the reduced wave equation $\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 + \partial^2 u/\partial z^2 + u = 0$ for which, uniformly in the spherical coordinates (θ, ϕ), $\int_0^\infty r^{1/2} u(x, y, z) dr = F(\theta, \phi)$, where the arbitrarily prescribed F satisfies a uniform Lipschitz condition with respect to spherical distance on the unit sphere. Furthermore an integral representation is given for u in terms of F. This result is parallel

to an earlier result of the author's relative to the reduced wave equation in E^2 [Trans. Amer. Math. Soc. **88** (1958), 388–399; MR **20** #1076].

R. W. McKelvey (Los Angeles, Calif.)

1748:

Grekov, A. V. The Dirichlet problem for certain quasi-linear parabolic equations. Dokl. Akad. Nauk SSSR **134** (1960), 255–258 (Russian); translated as Soviet Math. Dokl. **1** (1961), 1043–1047.

Let D denote a domain in the x, t plane bounded by segments in the horizontal lines $t=0$ and $t=T$ and the curves $x=\varphi_1(t)$ and $x=\varphi_2(t)$, where $\varphi_1(t)<\varphi_2(t)$ for $0\leq t\leq T$. Let Γ denote the sides and base of D. The author considers the Dirichlet problem for the quasilinear equation $u_t-a(x,t,u)u_{xx}+b(x,t,u)u_x+f(x,t,u)=0$ with $u|_\Gamma=0$, and lists hypotheses on a, b, f and φ_i under which existence and uniqueness of a solution is assured. To each function $v(x,t)$ a function $u(x,t)$, $u=u(v)$, is determined as a solution of the linear equation $u_t-a(x,t,v)u_{xx}+b(x,t,v)u_x+(u/v)\int_0^v \partial f(x,t,y)/\partial y\,dy+f(x,t,0)=0$, $u|_\Gamma=0$. The problem of existence is reduced to that of existence of a fixed point for $u=u(v)$. The latter is a consequence of Schauder's fixed point theorem, shown by applying estimates obtained by A. Friedman [J. Math. Mech. **7** (1958), 393–417, 771–792; MR **21** #7362, #7363] to solutions of the linear equations. Subsequent theorems come finally to a discussion of equations of the form $u_t=a(x,t,u,u_x)u_{xx}+f(x,t,u,u_x)$.

A. N. Milgram (Minneapolis, Minn.)

1749:

Krzyżański, M. Sur l'unicité des solutions des second et troisième problèmes de Fourier relatifs à l'équation linéaire normale du type parabolique. Ann. Polon. Math. **7** (1960), 201–208.

Consider the uniformly parabolic equation with bounded coefficients

$$(1)\quad \sum a_{ij}(x,t)\frac{\partial^2 u}{\partial x_i\partial x_j}+\sum b_i(x,t)\frac{\partial u}{\partial x_i}+c(x,t)u-\frac{\partial u}{\partial t}=f(x,t)$$

in a cylinder $E=\{(x,t)\colon x\in D,\ 0<t<T\}$, and the initial and boundary conditions (2) $u(x,0)=\varphi(x)$ for $x\in D$, (3) $\partial u/\partial l+h(x,t)u=g(x,t)$ for $x\in\partial D$, $0<t<T$, where ∂D is the boundary of D. Assume that $\cos(l,\nu)\geq\gamma_0>0$ where ν is the normal to ∂D. The author then proves two uniqueness theorems. Theorem 1: Assume that D is bounded and that there exists a function $G(x)$, C^2 in D and C^1 in $\overline{D}$, such that grad $G\neq 0$ on ∂D and $G=0$ defines ∂D. Then there exists at most one solution of (1)–(3) in E. Theorem 2: Assume that D is the exterior of a bounded domain, that ∂D is defined by $G=0$ and grad $G\neq 0$ on ∂D, where $G(x)$ is C^2 in D, C^1 in $\overline{D}$, and $G(x)=|x|$ for $|x|$ sufficiently large. Then there exists at most one solution of (1)–(3) in E which satisfies the growth condition $|u(x,t)|\leq M\times\exp\{K|x|^2\}$ for some $M>0$, $K>0$. The proofs involve the construction of comparison functions and an application of the maximum principle.

A. Friedman (Minneapolis, Minn.)

1750:

Čeremnyh, Yu. N. Asymptotic theory of the solutions of parabolic equations. Izv. Akad. Nauk SSSR. Ser. Mat. **23** (1959), 913–924. (Russian)

The equation

$$(*)\qquad a(x,t)u_{xx}+b(x,t)u_x+c(x,t)u=u_t$$

is considered in the region $G=\{(x,t)\,|\,0<x<\psi(t),\ t\geq 0\}$ under the assumptions that ψ is smooth, that $a,b,c\in C^{(0)}(G)$, and that $a(x,t)\geq a_0>0$, $c(x,t)\leq 0$ and $b(x,t)\leq B$ in G. The author modifies methods used by Krzyżański [Bull. Acad. Polon. Sci. Cl. III **4** (1956), 247–251; MR **18**, 47] to obtain uniform bounds for a solution of (*) in G which satisfies the boundary condition $u(0,t)=u(\psi,t)=0$ for $t\geq 0$. Theorem: If $\psi\in C^{(1)}(G)$, $0<\psi(t)\leq 1$, and $-B/(4H)\leq\psi'(t)\leq 0$ $(H=\exp(B/a_0))$, then

$$|u(x,t)|<2\max_{0<x<\psi(0)}|u(x,0)|\exp\{-a_0\int_0^t\psi^{-2}(\tau)d\tau/(2H)\}$$

in G. Under further conditions on ψ, on the coefficients of (*) and on the sign and order of magnitude of u, lower bounds for u are found. Corresponding results are derived in case u satisfies the more general boundary condition $u(0,t)=u(\psi,t)=\Phi(t)$ $(t\geq 0)$, where $\Phi\in C^{(1)}$, $\Phi(t)>0$ and $\Phi'(t)<0$ for $t\geq 0$.

N. D. Kazarinoff (Moscow)

1751:

Čžou, Yuĭ-lin'. Boundary problems for non-linear parabolic equations. Mat. Sb. (N.S.) **47 (89)** (1959), 431–484. (Russian)

The author proves in detail the results announced in Dokl. Akad. Nauk SSSR (N.S.) **117** (1957), 195–198 [MR **20** #3374]. In addition, the author studies the behavior as $t\to+\infty$ of the solution of the second boundary problem for

$$\sum_{i,j=1}^{N}a_{ij}(x)u_{,ij}+\sum_{i=1}^{N}b_i(x)u_{,i}+f(x,u)=u_{,t}.$$

D. G. Aronson (Minneapolis, Minn.)

1752:

Vitásek, Emil. Über die quasistationäre Lösung der Wärmeleitungsgleichung. Apl. Mat. **5** (1960), 109–140. (Czech and Russian summaries)

Consider the equation (1) $\partial u/\partial t=a^{-1}\partial^2 u/\partial x^2+q(t+kt_0)$, $kb<x<(k+1)b$ $(k=0,1,2,\cdots)$, $0<t<\infty$, where $q(t)$ is a given function and a, b are positive constants. Consider, further, the conditions (2) $u(x,0)=0$, $0<x<b$, $u(x,0)=f(x-b)$, $b<x<\infty$, and (3) $u(0,t)=0$, $0<t<\infty$. Set $(Af)(x)=u(x,t_0)$. We say that $f\in M$ if f is continuous for $0<x<\infty$ and if there exist constants α, b_0, b_1 such that (a) $0\leq f(x)\leq\alpha x+\alpha b$, (b) $\limsup_{h\to 0}[f(x+h)-f(x)]/h\leq b_0+b_1x$ and $\liminf_{h\to 0}[f(x+h)-f(x)]/h\geq-(b_0+b_1x)$ for all $0<x<\infty$. We say that there exists a quasi-stationary solution of (1), (2), (3) in a set B if there exists $f\in B$ such that $(Af)(x)=f(x)$ for $0<x<\infty$. Theorem 1: If $q(t)$ is continuous, non-negative and bounded for $0<t<\infty$, then there exists a unique quasi-stationary solution in the set M. Theorem 2: Let $f_0\in M$, $f_{k+1}=Af_k$. Then $\{f_k(x)\}$ converges uniformly in bounded intervals to the quasi-stationary solution of the previous theorem. Theorem 3: If, in addition to the assumptions of theorem 1, $q(t)$ is monotone decreasing and $\int^\infty q(t)dt<\infty$, then the solution is bounded.

A. Friedman (Minneapolis, Minn.)

1753:

Szeptycki, P. Existence theorem for the first boundary problem for a quasilinear elliptic system. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **7** (1959), 419–424. (Russian summary, unbound insert)

Consider the elliptic system

$$(1)\quad \sum a_{ij}(x, u, \operatorname{grad} u)\frac{\partial^2 u_j}{\partial x_i \partial x_j} + \sum a_i(x, u, \operatorname{grad} u)\frac{\partial u_j}{\partial x_i} + \sum c_{jk}(x, u, \operatorname{grad} u)u_k = 0$$

in a bounded N-dimensional domain D, and the boundary conditions (2) $u_l = \varphi_l$ on the boundary ∂D. The coefficients of (1) are assumed to be Hölder continuous, φ_l and ∂D are of class $C^{2+\alpha}$ (for some $\alpha > 0$), (c_{kl}) is a negative definite matrix, and finally the a_{ij} satisfy a condition somewhat stronger than the K_σ'-condition of H. O. Cordes [Math. Ann. **131** (1956), 278–312; MR **19**, 961]. The author first derives a maximum principle for solutions of (1), (2), and then uses it together with a priori estimates of Schauder and Cordes to establish the existence of a solution of (1), (2), employing Schauder's fixed-point theorem.

A. Friedman (Minneapolis, Minn.)

1754:

Dezin, A. A. A special system of equations. Dokl. Akad. Nauk SSSR **127** (1959), 497–500. (Russian)

The author introduces and studies (using the method of orthogonal projections) some systems of linear partial differential equations (v (≤ 4) variables, M equations, M unknown functions) which generalize the Cauchy-Riemann equations.

C. Foiaș (Bucharest)

1755:

Ždanovič, V. F. Solution by the Fourier method of non-selfadjoint mixed problems for hyperbolic systems in the plane. II. Mat. Sb. (N.S.) **48 (90)** (1959), 447–498. (Russian)

[For part I, see Mat. Sb. (N.S.) **47 (89)** (1959), 307–354; MR **21** #1439.] In part II the author studies the operator $L_0 : L_0 y(x) = A(x)y'(x) + B(x)y(x)$ which operates on those functions $y(x) \in C^{(1)}(0, l)$ that satisfy the boundary conditions

$$MA(0)y'(0) + [MB(0) + N]y(0) + PA(l)y'(l) + [PB(l) + Q]y(l) = 0.$$

[Cf. part I.] First, using the Birkhoff-Langer method, he affirms the possibility of constructing for the system $L_0 y(x) = \lambda y(x)$ a fundamental matrix $Y(x, \lambda)$ having certain continuity and analyticity properties and gives its representation. This becomes a tool for the study of the distribution of zeros of the characteristic determinant $\Delta(\lambda)$. "Regularity" of the boundary conditions is defined in such a way that for regular boundary conditions the behavior of the zeros of $\Delta(\lambda)$ as $\lambda \to \infty$ is determined by a certain Dirichlet polynomial, and one is led to a representation of the eigenvalues of L_0 in terms of an expansion of this polynomial.

Next he proves the existence of a Green's function for the operator $L_0 - \lambda E$ and deduces a number of its properties. Finally he proves an expansion theorem for functions $f(x)$ in the domain of L_0 and proves convergence in a Banach space $D_2(0, l)$ with norm given by

$$\|f(x)\|^2_{D_2(0,l)} = \|f(x)\|^2_{L_2(0,l)} + \|Mf(0) + Pf(l)\|^2.$$

R. N. Goss (San Diego, Calif.)

1756:

Ždanovič, V. F. Solution by the Fourier method of non-selfadjoint mixed problems for hyperbolic systems in a plane. III. Mat. Sb. (N.S.) **49 (91)** (1959), 233–266. (Russian)

In part III one returns to the formal series solution obtained in part I for the purpose of establishing its validity. Convergence is proved with respect to a norm in a Banach space $M_2(\Omega)$, $\Omega = [0 \leq x \leq l;\ 0 \leq t \leq T]$, derived from the $D_2(0, l)$ norm, under the assumption that the boundary conditions are regular. Next a classical solution of the mixed boundary-value problem of part I is defined in the usual way, and the formal series is shown to be a classical solution provided that the boundary conditions are regular and $f(x)$ and its first two derivatives satisfy conditions guaranteeing uniform convergence of the series itself and those obtained by differentiation. In the last section the series is shown to converge to a generalized solution under the sole requirement of regularity of the boundary conditions.

R. N. Goss (San Diego, Calif.)

1757:

Zerner, Martin. Sur le support de la solution d'un problème de Cauchy. C. R. Acad. Sci. Paris **250** (1960), 802–804.

Let $a(\partial/\partial x_1, \cdots, \partial/\partial x_n)$ be a totally hyperbolic operator which is also irreducible. Let $\dot{C}(a)$ be the inside of the convex cone in $x_1 > 0$ enveloped by all space-like hyperplanes through the origin. $\dot{C}$ is assumed not empty. Let v be a distribution with compact support $S(v)$. Let u be the solution of $a(\partial/\partial x)u = v$ which vanishes for $x_1 < k$ for some k. Let $I(v)$ be the intersection of all cones like $\dot{C}$ with vertex in $S(v)$. If the support of u is not compact it meets every space-like $n-2$-plane which meets $I(v)$.

P. Ungar (New York)

1758:

Ikebe, Nobunori. The fundamental solution and the uniqueness theorem of Cauchy problem for the linear parabolic system. Comment. Math. Univ. St. Paul. **8** (1960), 71–76.

Construction d'une solution fondamentale pour un système parabolique au sens de Petrowsky, à coefficients dépendant des variables d'espace et de temps. Variantes de résultats de S. D. Eĭdelman [Mat. Sb. (N.S.) **38 (80)** (1956), 51–92; Dokl. Akad. Nauk. SSSR **98** (1954), 913–915; MR **17**, 857].

J. L. Lions (Nancy)

1759:

Žitomirskiĭ, Ya. I. The Cauchy problem for certain types of systems, parabolic in the sense of G. E. Šilov, of linear partial differential equations with variable coefficients. Izv. Akad. Nauk SSSR. Ser. Mat. **23** (1959), 925–932. (Russian)

On considère le système

(*) $\partial u/\partial t - P(x, i^{-1}\partial/\partial x)u = 0,\ x \in R^n,\ u = \{u_1, \cdots, u_N\}$; si $P(x, i^{-1}\partial/\partial x) = P_0(i^{-1}\partial/\partial x) + P_1(x, i^{-1}\partial/\partial x)$, on suppose que $\partial/\partial t - P_0$ est parabolique au sens de Šilov [cf. Šilov, Uspehi Mat. Nauk **10** (1955), no. 4 (66), 89–100; MR **17**, 495; Gel'fand et Šilov, *Nekotorye voprosy teorii differencial'nyh uravneniĭ*, Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, 1958; MR **21** #5142b] et que P_1 est d'ordre "assez

petit" (cf. (10), (10′) chez l'A.). Alors l'A. résout le problème de Cauchy pour (*); à l'aide de majorations convenables (No. 2) sur la solution élémentaire de $\partial/\partial t - P_0$ on transforme le problème en un système d'équations intégrales, relevant de la méthode de approximations successives. *J. L. Lions* (Nancy)

1760:

Hörmander, Lars. Null solutions of partial differential equations. Arch. Rational Mech. Anal. **4**, 255–261 (1960).

Let $P(D)$ $(D_k = \partial/i\partial x_k)$ be a linear partial differential operator with constant coefficients and let P_0 be its principal part. A regular solution u of $Pu = 0$ is called a null solution provided u vanishes outside a half-space $H: x\xi > 0$. The author has shown that non-vanishing null solutions exist if and only if the plane $x\xi = 0$ is characteristic for P, $P_0(\xi) = 0$ [Acta Math. **94** (1955), 161–248; MR **17**, 853]. He now shows that the null solutions span all solutions of $Pu = 0$ which are regular in H provided ξ is characteristic and P is irreducible. More generally, the span is all solutions in H of $P_1(D)u = 0$ where $P = P_1P_2$, ξ is characteristic for all irreducible factors of P_1 and non-characteristic for P_2. *L. Gårding* (Lund)

1761:

Bastiani, Andrée. Méthode pratique de résolution du problème de Cauchy mixte. C. R. Acad. Sci. Paris **250** (1960), 260–262.

On considère un système S d'équations aux dérivées partielles et à coefficients constants, dont les fonctions inconnues sont de la forme $u(x, t)$, $x \in R^n$, $t \in R$, et l'on se propose de trouver les solutions qui satisfont aux conditions initiales de Cauchy (pour $t = 0$) et aux conditions à la frontière pour $t > 0$. Ces conditions, complétées par des fonctions arbitraires Φ, sont interprétées comme partie droite du système S, considéré comme système en distributions sur R^{n+1}. Si le problème en distributions a des solutions, il reste à déterminer les fonctions Φ, afin que ces solutions deviennent les solutions du problème initiale. L'A. montre comment on peut résoudre, de cette manière, le premier problème de Burmister de la théorie de l'élasticité. *G. Marinescu* (Bucharest)

1762:

Cinquini-Cibrario, Maria. Teoremi di unicità per sistemi di equazioni a derivate parziali in più variabili indipendenti. Ann. Mat. Pura Appl. (4) **48** (1959), 103–134.

Riassunto dell'autore: "Sotto ipotesi molto ampie sono dimostrati alcuni teoremi di unicità e di dipendenza continua dai dati della soluzione del problema di Cauchy (in senso generalizzato) del sistema

$$\text{(I)}\quad \sum_{j=1}^{m} A_{ij}(x, y_1, \cdots, y_h; z_1, \cdots, z_m)\left\{\frac{\partial z_j(x, y_1, \cdots, y_h)}{\partial x} + \sum_{r=1}^{h} \rho_{ir}(x, y_1, \cdots, y_h; z_1, \cdots, z_m)\frac{\partial z_j(x, y_1, \cdots, y_h)}{\partial y_r}\right\} = f_i(x, y_1, \cdots, y_h; z_1, \cdots, z_m) \qquad (i = 1, 2, \cdots, m).$$

La soluzione è ricercata nel campo funzionale, costituito dai sistemi di funzioni $z_i(x, y_1, \cdots, y_h)$ $(i = 1, 2, \cdots, m)$, che sono definite in un ben determinato campo, sono ivi assolutamente continue in x e lipschitziane in $y_1, \cdots, y_h$, e soddisfano il sistema (I) quasi ovunque in tale campo."

1763:

Sobolevskiĭ, P. E. Non-stationary equations of viscous fluid dynamics. Dokl. Akad. Nauk SSSR **128** (1959), 45–48. (Russian)

Many kinds of generalized solutions have been defined relative to problems of the type

$$\text{(1)}\quad U_t' - \nu\Delta U + u_k U_{x_k}' = \operatorname{grad} p(t, x) + f(t, x), \qquad \operatorname{div} U = 0 \quad (0 < t \leq T, x \text{ in } \Omega),$$

where Ω is an open region in E^m, where $\Gamma = \partial\Omega$ and where $U = (u_k)$, U_0 and f are vector functions defined on $\Omega + \Gamma$. S. G. Kreĭn [Uspehi Mat. Nauk **12** (1957), no. 1 (73), 208–211; MR **19**, 36] calls a solution of the problem

$$\text{(2)}\quad U_t' + \nu AU + Pu_k U_{x_k}' = Pf, \quad U(0) = U_0,$$

a generalized solution of (1). Here P is a projection operator on $\mathscr{L}_2(\Omega)$ into H, where H is the closure of a set of smooth solenoidal vectors which vanish on Γ; A is a Friedrichs self-adjoint extension of $-P\Delta$ to H, $P\Delta$ being initially defined on $H \cap \mathring{W}_2^2$.

M. A. Krasnosel'skiĭ, S. G. Kreĭn and the author [same Dokl. **112** (1957), 990–993; MR **19**, 747; Sobolevskiĭ, ibid. **116** (1957), 754–757; MR **20** #4194] established an existence-uniqueness theorem for weak solutions, in the above sense, of (1) for U_0 in $H \cap \mathring{W}_2^2$ and f continuously differentiable in t, provided $m = 2$ or 3. Now the author demonstrates existence and uniqueness of an even weaker solution. Theorem 1: If U_0 lies in $D(A^{1/2})$ and if f is in $\mathscr{L}_2(0, T)$, then on some interval $[0, t_0]$ there exists a unique solution to the problem

$$\text{(3)}\quad (A^{-\alpha}U)_t' + A^{1-\alpha}U + \overline{A^{-\alpha}P}u_k U_{x_k}' = A^{-\alpha}Pf, \qquad U(0) = U_0, \quad \alpha = 2^{-n} \quad (n = 1, 2, \cdots);$$

$A^{1/2}U(t)$ is continuous and $A^{-\alpha}U(t)$ is absolutely continuous in H; $\|(A^{-\alpha}U)_t'\|$ and $\|A^{1-\alpha}U\|$ are square integrable; for any V in $H \cap \mathring{W}_2^1$,

$$\left(\int_\Omega UV\,dx\right)_t' + \nu\int_\Omega U_{x_k}' V_{x_k}\,dx + \int_\Omega u_k U_{x_k}' V\,dx = \int_\Omega fV\,dx.$$

Theorem 2: If $m = 2$, then the solution of (3) exists for all $t \geqq 0$. This result improves one of Ladyženskaya's [Dokl. Akad. Nauk SSSR **123** (1958), 427–429; MR **21** #7675].

The proofs are based on an integral representation for $(-\Delta)^{-\alpha}V(x)$ $(0 < \alpha < m/2)$ and the inequality

$$\|A^\alpha U\| \geq \|(-\Delta)^\alpha U\|, \quad U \text{ in } D(A^\alpha).$$

An estimate of $\|U^{(1)}(t) - U^{(2)}(t)\|$ is given for functions $U^{(1)}$ and $U^{(2)}$ which are solutions of equations of type (2) corresponding to different right members and to different initial data. A theorem describing the stability of solutions of (2) as $t \to \infty$ is also given. *N. D. Kazarinoff* (Moscow)

1764:

Mitropolsky, Ju. A. Sur les multiplicités intégrales des systèmes d'équations différentielles non linéaires ayant un petit paramètre. Ann. Mat. Pura Appl. (4) **49** (1960), 181–192.

This paper is a good summary of the types of results that have been obtained by the Russian school of Bogolyubov, Mitropol'skiĭ and Lykova on the existence of

integral manifolds of nonlinear differential equations containing a small parameter. The original papers have been previously reviewed [see N. Bogolyubov and Yu. Mitropol'skiĭ, *Asymptotic methods in the theory of nonlinear oscillations*, Gosudarstv. Izdat. Fis.-Mat. Lit., Moscow, 1958; 1st ed., 1955; MR **20** #6812; **17**, 368; Yu. Mitropol'skiĭ, *Transient processes in nonlinear oscillatory systems*, Izdat. Akad. Nauk. Ukrain. SSR, Kiev, 1955; MR **17**, 735; O. Lykova, Ukrain. Mat. Ž. **9** (1957), 419–431; MR **19**, 857]. (Also see Mitropol'skiĭ, Ukrain. Mat. Ž. **10** (1958), 270–279, 389–393 [MR **21** #176a, b].)

J. K. Hale (Baltimore, Md.)

POTENTIAL THEORY

1765:

Nikol'skiĭ, S. M. On an estimate of a function with finite Dirichlet integral and an application to boundary problems. Izv. Akad. Nauk SSSR. Ser. Mat. **23** (1959), 677–696. (Russian)

In n-space, let Q be a hyperplane normal to a coordinate axis, T a strip of width h between two such hyperplanes normal to Q, I an interval meeting the three hyperplanes in question. The intersection H of Q, T and I may be termed sectional strip of width h of I. Further, for appropriate functions f and sets E, let $S_n(f, E)$ and $D_n(f, E)$ denote the n-ple integrals over E of the quantities f^2 and (grad f)^2, respectively. The author establishes an inequality of the type

$$S_{n-1}(f, H) \leq C\{S_n(f, I) + D_n(f, I)\}h \log (1/h)$$

for almost all sectional strips H, where C depends only on I. The applications include a criterion for a harmonic function u to be the solution of certain boundary-value problems.

L. C. Young (Bloomington, Ind.)

1766:

Evgrafov, M. A.; Čegis, I. A. Generalization of the Phragmén-Lindelöf theorem on analytic functions to harmonic functions in space. Dokl. Akad. Nauk SSSR **134** (1960), 259–262 (Russian); translated as Soviet Math. Dokl. **1** (1961), 1047–1051.

The authors prove the following two theorems on harmonic functions in space. Theorem 1: Let $u(r, \varphi, x)$ be harmonic in the cylinder $r \leq a$, $0 \leq \varphi < 2\pi$, $-\infty < x < \infty$. If $u(a, \varphi, x) = 0$, $|\partial u(a, \varphi, x)/\partial n| < C$, and

$$\max_{(r,\varphi)} |u(r, \varphi, x)| < C \exp e^{\pi|x|/(2+\varepsilon)a} \quad (\varepsilon > 0),$$

then $u(r, \varphi, x) \equiv 0$. Theorem 2: Let $u(r, \theta, \varphi)$ be harmonic in the cone $0 < r < \infty$, $0 \leq \theta \leq \theta_0 < \pi$. If $u(r, \theta_0, \varphi) = 0$, $|\partial u(r, \theta_0, \varphi)/\partial\theta| < C$, and

$$\max_{(\theta,\varphi)} |u(r, \theta, \varphi)| < C \exp (r + (1/r))^{\pi/2\theta_0 - \varepsilon} \quad (\varepsilon > 0),$$

then $u(r, \theta, \varphi) \equiv 0$.

The proofs are based on the following uniqueness theorem for Dirichlet series. Theorem 3: Let $F(z) = \sum_{n=1}^{\infty} a_n e^{\lambda_n z}$ be an entire function, and let $|a_n|^{1/\lambda_n} < Cn^{-2-\varepsilon}$ $(\varepsilon > 0)$; $\lim n/\lambda_n = \alpha$, $0 < \alpha < \infty$, $\lambda_n > 0$. If in addition $|F(x)| < C$, $-\infty < x < \infty$, then $F(z) \equiv 0$. Complete proofs are given for all three theorems.

A. L. Shields (New York)

1767:

Kishi, Masanori. Capacitability of analytic sets. Nagoya Math. J. **16** (1960), 91–109.

Let Ω be a locally compact separable metric space and Φ be a positive symmetric kernel. A measure will always mean a non-negative measure and the potential of a measure μ with the kernel Φ will be denoted by $U^{\mu}(P)$. Consider a functional c, defined on the space of measures in Ω, which satisfies the following postulates: $0 \leq c(\mu) \leq \infty$, $c(0) = 0$, $c(\mu + \nu) \leq c(\mu) + c(\nu)$, $c(t\mu) = tc(\mu)$ if $t \geq 0$, lower semicontinuity with respect to the value topology. A set X is called polar if there is a measure μ such that $U^{\mu}(P) = \infty$ on X; and a class $\mathfrak{F}_X$ is defined by all measures whose potentials are ≥ 1 on X except on polar sets. First, $c(X)$ is defined by $\inf_{\mu \in \mathfrak{F}_X} c(\mu)$; if $\mathfrak{F}_X$ is empty, $c(X)$ is set equal to ∞. Next, $\mathrm{cap}_i(X) = \sup c(K)$ where $K \subset X$ is a compact set and $\mathrm{cap}_e(X) = \inf \mathrm{cap}_i(G)$ where $G \supset X$ is an open set. X is called capacitable if $\mathrm{cap}_i(X) = \mathrm{cap}_e(X)$. Assume that K is not a polar set if and only if there is $\mu \neq 0$ with $S_\mu \subset K$ such that $U^{\mu}(P)$ is continuous in Ω; that every potential is quasi-continuous in Ω, that is, for any $\varepsilon > 0$, there is an open set G_ε such that $\mathrm{cap}_i(G_\varepsilon) < \varepsilon$ and the restriction of U^{μ} to $\Omega - G_\varepsilon$ is finite and continuous; and that, if $c(X) \leq M$, then there exists $\mu_n \in \mathfrak{F}_{X_n}$ such that $c(\mu_n)$ is sufficiently near $c(X_n)$ and $\mu_n(\Omega)$ is bounded by a constant depending only on M. When all these assumptions are satisfied, we shall say that the capacity is admissible. We assume also that, for any compact set K and $\varepsilon > 0$, there is a compact set L such that $\Phi(P, Q) < \varepsilon$ on $K \times (\Omega - L)$. The main theorem is the following: If the capacity is admissible, every analytic set is capacitable, where an analytic set is defined as a continuous image of a $K_{\sigma\delta}$ set contained in a compact space. Two examples of c are particularly studied. The first $c(\mu)$ is defined by $\mu(\Omega)$, and secondly the energy (μ, μ) is taken for $c^2(\mu)$ under the assumption that the kernel is of positive type (= positive definite). The results for the first kind of capacity were announced by the author in Proc. Japan Acad. **35** (1959), 158–160 [MR **21** #7372].

M. Ohtsuka (Hiroshima)

1768:

Nečas, Jindřich. L'extension de l'espace des conditions aux limites du problème biharmonique pour les domaines à points anguleux. Czechoslovak Math. J. **9** (**84**) (1959), 339–371. (Russian summary)

Let Ω be a simply-connected plane region whose boundary C has a finite number of corner points. Two functions f_x, f_y of arc length on C are given and it is required to find a function u biharmonic in Ω such that on C, $\partial u/\partial x = f_x$, $\partial u/\partial y = f_y$. The author employs methods of functional analysis, in particular, imbedding theorems of Sobolev [*Nekotorye primeneniya funkcional'nogo analiza v matematičeskoĭ fizike*, Izdat. Leningrad. Gos. Univ., Leningrad, 1950; MR **14**, 565], in conjunction with his earlier work on the biharmonic problem for convex and non-convex infinite wedges [Časopis Pešt. Mat. **83** (1958), 257–286, 399–424; **84** (1959), 90–98; MR **21** #1462; **21** #5826], to show that $f_x, f_y \in L_p(C)$, where $\int_C f_x dx + f_y dy = 0$, are allowable boundary conditions whenever $p > 1/(1+\mu)$, where $-\frac{1}{2} < \mu \leq 0$ and μ is a number depending upon the corner points of C; in particular, boundary functions f_x, $f_y \in L_2(C)$ are included. If Ω is convex any $p > 1$ is allowable.

J. F. Heyda (Cincinnati, Ohio)

FINITE DIFFERENCES AND FUNCTIONAL EQUATIONS

See also 1673.

1769:

Feldmann, L. On linear difference equations with constant coefficients. Period. Polytech. Elec. Engrg. **3** (1959), 247–257.

The author sets up an operational technique for solving linear difference equations with constant coefficients. He believes his method simpler than those already in the literature. *T. Fort* (Coral Cables, Fla.)

1770:

Perron, Oskar. Über lineare Differenzengleichungen und eine Anwendung auf lineare Differentialgleichungen mit Polynomkoeffizienten. Math. Z. **72** (1959/60), 16–24.

For difference equations of the form

$$D_{j+m}+p_1(j)D_{j+m-1}+\cdots+p_m(j)D_j=0 \qquad (j=0,1,2,\cdots), \tag{1}$$

where $p_i(j)\sim A_i j^{k(i)}$ $(i=1,2,\cdots,m)$ with A_i complex, $k(i)$ real, the author proves a theorem on the existence of a fundamental system $D_j^{(1)},\cdots,D_j^{(m)}$ of solutions such that the "orders"

$$\rho(s)=-\limsup_{j\to\infty}\{[\log(j!)]^{-1}[\log|D_j(s)|]\}$$

and the corresponding "types"

$$\sigma(s)=\limsup_{j\to\infty}[|D_j^{(s)}|(j!)^{\rho(s)}]^{1/j}$$

assume a pattern of values provided by a Newton polygon constructed on the points $(i, k(i))$. The result with respect to the $\rho(s)$ is a special case of a result due to the author [Acta Math. **34** (1911), 109–137]; the result with respect to the $\sigma(s)$ was found by Kreuser [Diss., Tubingen, 1914]. The important contribution of the present paper is to provide a considerably simplified method of proof. This is done by deriving the results as direct consequences of a well-known theorem of the author on sum equations $\sum_{n=0}^{\infty}a_{jn}x_{j+n}=0$ $(j=0,1,\cdots)$ [Math. Ann. **84** (1921), 1–15]. The theorem on difference equations is then applied to the study of the order and corresponding type of the (necessarily entire) solutions $y(x)=\sum_{j=0}^{\infty}D_jx^j$ of ordinary differential equations of the form $d^my/dx^m+P_1(x)\times d^{m-1}y/dx^{m-1}+\cdots+P_m(x)y=H(x)$, where H and the P_i are polynomials. *W. Strodt* (New York)

1771:

Torng, H. C. Second-order non-linear difference equations containing small parameters. J. Franklin Inst. **269** (1960), 97–104.

In this paper the author studies approximate solution of the equations

$$X_{n+2}+A_1X_{n+1}+A_2X_n=\mu f(X_n,X_{n+1}),$$

where μ is small and A_1 and A_2 are constants. His method could be used for equations of higher order. In brief he utilizes the Lagrange method of variation of parameters and well-known formulas for the expansion of a function of an integral argument in terms of sines and cosines. His procedure parallels frequently-used procedures for the approximate solution of differential equations.

T. Fort (Coral Gables, Fla.)

1772:

Climescu, Al. Résolution d'un système d'équations fonctionnelles. Bul. Inst. Politehn. Iaşi (N.S.) **4** (8) (1958), no. 3–4, 7–10. (Romanian. Russian and French summaries)

The author solves the system

$$\begin{aligned} &a_0(x)a_0(y)=a_0(xy),\\ &a_0(x)a_1(y)+a_1(x)a_0(y)=a_1(xy),\\ &\cdots\qquad\qquad\cdots\\ &a_0(x)a_n(y)+a_1(x)a_{n-1}(y)+\cdots+a_n(x)a_0(y)=a_n(xy). \end{aligned}$$

The only real solutions, continuous except perhaps at 0, have one of the forms $a_p(x)\equiv 0$; or $a_0(x)\equiv 1$, $a_p(x)\equiv 0$ $(p\geq 1)$; or $a_p(x)=a_0(x)b_p(x)$, $a_p(0)=0$, $a_0(x)$ of one of the forms $a_0(x)=|x|^c$ and $a_0(x)=x|x|^c$, $b_p(0)=0$, $b_p(x)=c_p(\log|x|)$, where $c_p(u)$ are the coefficients of the power series of $\exp\{u(\alpha_1z+\cdots+\alpha_nz^n)\}$, with arbitrary α's.

R. P. Boas, Jr. (Evanston, Ill.)

SEQUENCES, SERIES, SUMMABILITY

See also 1804.

1773:

Climescu, Al. Une classe d'inégalités. Bul. Inst. Politehn. Iaşi (N.S.) **4** (8) (1958), no. 3–4, 1–4. (Romanian. Russian and French summaries)

The author bases an inductive proof of the inequality between the arithmetic and geometric means on the following inequality: for $a\geq 0$ and $b\geq 0$ we have $na^{n+1}-(n+1)a^nb+b^{n+1}\geq 0$ with equality if and only if $a=b$. He uses the same idea to establish an inequality of very general appearance. *R. P. Boas, Jr.* (Evanston, Ill.)

1774:

Cugiani, Marco. Sulla densità massimale delle successioni e sui teoremi di Fabry e Pólya-Bieberbach. Ist. Lombardo Accad. Sci. Lett. Rend. A **93** (1959), 273–286.

The author gives a new definition of the Pólya maximum density of an increasing sequence of real numbers s_k that satisfy the separation condition $s_{k+1}-s_k>l>0$. By using this he shows the equivalence of two apparently different theorems on singular points of power series. The definition reads as follows. Choose a sequence of integers $\{n_k\}$ and positive numbers $\theta^*<\frac{1}{2}$, θ^{**} so that we can construct a sequence of disjoint half-open intervals $I_h=((1-\theta_h')n_h, (1+\theta_h'')n_h]$, with $\theta^*\leq\theta_h'\leq 1-\theta^*$, $\theta^*\leq\theta_h''\leq\theta^{**}$, such that every s_k is in some I_h. Subdivide each I_h into nonoverlapping and exhaustive subintervals $J_{h,\nu}$ of lengths $j_{h,\nu}$, put $j_h=\max_\nu j_{h,\nu}$, and let $j_h=o(n_h)$. Now let $N_{h,\nu}$ be the number of s_k in $J_{h,\nu}$ and put $d_{h,\nu}=N_{h,\nu}/j_{h,\nu}$. Say that $\{s_k\}$ has property $F(\Delta_0,\{I_h\})$ if there is a sub-division $J_{h,\nu}$ such that $d_{h,\nu}\leq\Delta_0$ for sufficiently large h. Then the maximum density of $\{s_k\}$ turns out to be $\inf\Delta_0$ for all Δ_0 such that $\{s_k\}$ has property $F(\Delta_0,\{I_h\})$. The theorems on singular points can be found, for example, in Bieberbach, *Analytische Fortsetzung*, Ergebnisse der Math. (N.F.) no. 3, Springer, Berlin, 1955 [MR 16, 913], pp. 43 ff.

R. P. Boas, Jr. (Evanston, Ill.)

1775:

Macphail, M. S. Sequence and series transformations. Canad. Math. Bull. **3** (1960), 80–83.

The author considers the summability methods A: $t_n=\sum_{k=0}^{\infty} a_{n,k}s_k$, and B: $T_n=\sum_{k=0}^{\infty} b_{n,k}u_k$, where $b_{n,k}=a_{n,k}+a_{n,k+1}+\cdots$; these are the sequence-to-sequence and series-to-sequence transformations by the same method, and if $s_k=u_0+\cdots+u_k$, the series $\sum u_k$ and the sequence $\{s_k\}$ are said to be summable B and A respectively. The matrix A is assumed to be regular, i.e., a T-matrix. The object of this note is to give simple sufficient conditions for $B\supset A$ or $A\subset B$ depending on the matrix elements $b_{n,k}$ only. Theorem 1: In order that $B\supset A$, it is sufficient that for each $n=0, 1, \cdots$ there is a positive constant R_n such that $|1-b_{n,k+1}/b_{n,k}|>R_n$ for every k. {The author omits to add as an extra condition that all the leading diagonal elements of A_n, i.e., $a_{n,i}$ ($i=0, 1, 2, \cdots$), must be non-zero; for otherwise the reciprocal A_n^{-1} used in the proof does not exist.} Theorem 2: In order that $A\supset B$, it is sufficient that for each $n=0, 1, \cdots$ there is a constant M_n such that

$$|b_{n,k+1}|\sum_{r=0}^{k}|b_{n,r+1}^{-1}-b_{n,r}^{-1}|<M_n \quad (k=0, 1, \cdots),$$

and $\lim_{k\to\infty} b_{n,k}=0$. Theorem 3: In order that $A\supset B$, it is sufficient that for each $n=0, 1, \cdots$ there is a constant C_n ($0<C_n<1$) such that $|b_{n,k+1}/b_{n,k}|<C_n$ ($k=0, 1, \cdots$). For real $b_{n,k}$, it is sufficient that for each n, $b_{n,k}\to 0$ monotonically from a certain k on.

{On p. 81, 10 lines up from the bottom of the page, "kth row" should be "$(k+1)$th row."}

R. G. Cooke (London)

1776:

Wells, J. H. Hausdorff transforms of bounded sequences. Proc. Amer. Math. Soc. **11** (1960), 84–86.

Let ϕ be a complex-valued function of bounded variation in $[0, 1]$, such that $\phi(0)=0=\phi(+0)$ and $\phi(1)=1$, so that the Hausdorff transformation $H(\phi)$ defined by $y=H(\phi)x$ where

$$y_n=\int_0^1\sum_{p=0}^{n}C_{n,p}t^p(1-t)^{n-p}x_p\,d\phi(t)$$

is regular. The author proves that for each bounded sequence x of complex numbers, the set of limit points of $\{y_n\}$ is connected if and only if $\phi(1-)=\phi(1)$.

M. S. Ramanujan (Ann Arbor, Mich.)

1777:

Moser, Donald E. The Taylor and other methods of summability. Proc. Amer. Math. Soc. **11** (1960), 90–96.

The Taylor methods of summability are defined by the matrices

$$T(r)\equiv T_{nk}(r)=(1-r)^{n+1}C_{k,n}r^{k-n}, \quad r\neq 1,$$

and the methods $S(q)$ by the matrices

$$S(q)\equiv S_{nk}(q)=(1-q)^{n+1}C_{n+k,n}q^k, \quad q\neq 1.$$

The binary method Y transforms a sequence S_n into $y_n=\frac{1}{2}(S_{n-1}+S_n)$. The author compares the methods $T(r)$ and $S(q)$ with Y and the Cesàro method $(C, 1)$ and proves that for $0<r<1$, $T(r)\supset Y$ and $T(r)\not\supset (C, 1)$; for $0<q<1$, $S(q)\supset Y$ and $S(q)\not\supset (C, 1)$. The domains in which $\sum Z^n$ and $\sum a_nZ^n$ are summable by the iterated product method $B\cdot T(r)$ or $B\cdot S(q)$ are also determined.

M. S. Ramanujan (Ann Arbor, Mich.)

1778:

Petersen, G. M. Summability methods and unbounded sequences. Math. Scand. **7** (1959), 170–176.

Let A be a regular (Toeplitz) matrix satisfying the condition

$$(*) \qquad a_{mn}=0 \quad (n>\lambda(m)) \quad \text{and} \quad a_{m,\lambda(m)}\neq 0.$$

The author proves that to a regular matrix A which satisfies the condition (*) there corresponds a regular matrix B which satisfies (*) and is strictly stronger than A, so that every matrix which is stronger than A and sums a B-summable sequence which is not A-summable is stronger than B. The proof requires the lemma: If A is a regular matrix satisfying (*) above and $\lambda(m)=m$, $a_{mn}=0$ for $n<m-1$ and $|a_{m,m-1}a_{mm}^{-1}|\geq K>1$, $m\geq 2$, then every A-summable sequence has the form $\{s_m\}=\{cs_m'+\sigma_m\}$ where s_m' is an unbounded sequence, c is a constant and $\{\sigma_m\}$ is convergent. In theorem 2 he proves that if A is a regular matrix satisfying (*) and B is a regular matrix stronger than A and such that there exist two B-summable sequences none of whose (non-trivial) linear combinations is A-summable, then there exists a matrix C strictly stronger than A and with B strictly stronger than C. This is the analogue of a similar result of Brudno for bounded sequences [Mat. Sb. (N.S.) **16** (**58**) (1945), 191–247; MR **7**, 12; also, Peterson, Math. Z. **65** (1956), 494–496; MR **18**, 573].

M. S. Ramanujan (Ann Arbor, Mich.)

1779:

Meynart, C. ★Les séries et leur application à la résolution de divers problèmes pratiques d'analyse mathématique. Tome 1. Eyrolles, Paris, 1959. 198 pp. (6 inserts) 2800 F.

This is a book written for engineers by a practicing engineer. The aim of the book is to show that by a judicious use of series one can solve practically all problems of analysis. The book contains many examples demonstrating this procedure in a formal fashion. There is no discussion of the limitations of the method nor of its efficiency. Almost all series used are Maclaurin's series.

The first two brief chapters explain the purpose of the book and indicate what is to follow. The first part of chapter III gives without proof some of the elementary convergence tests together with theorems concerning operations with series. Taylor's and Maclaurin's series are discussed. The remainder of chapter III discusses summation of series by recognition of standard forms and also by numerical evaluation of the terms. A table giving the expansions in power series of many common functions concludes the chapter. Chapter IV deals with the algebra of series and derives in great detail formulas for multiplying and dividing series, raising then to powers, finding functions of series, differentiating and integrating them. Chapter V gives a number of examples of integral evaluation by termwise integration of power series. Chapter VI presents the method of power series as a general method for the solution of practically all differential equations. There is an extremely large number of examples worked out in detail.

Volume 2 is to include chapters on series solutions of differential systems, on operational calculus, on practical applications including electricity and ballistics, and on high-speed automatic computers.

J. G. Herriot (Stanford, Calif.)

APPROXIMATIONS AND EXPANSIONS

See also 1549.

1780:

Arsac, J.; Simon, J.-C. Représentation d'un phénomène physique par des sommes de translatées. Ann. Radioélec. **15** (1960), 217–227. (English and German summaries)

Authors' summary: "For the representation of a law, physicists commonly use sums of simple functions $\sum a_k g_k(x)$. Among these, the sums obtained by 'translating' a function $g(x)$ provide an exact or approximate representation of a function $f(x)$ with a bounded spectrum, $f(x)=\sum f(p\alpha)g((x/\alpha)-p)$. The coefficients are just the values of the function to be represented at evenly spaced points. A theory of the exact representation of a bounded spectrum function by a sum of translates with bounded spectra and a theory of the approximate representation of a bounded spectrum function by a sum of bounded support translates, allows of a generalisation of the Shannon representation, of obtaining many polynomial interpolation formulae, and of securing simple representations leading to very close approximations.

"Such methods are applied to the theory of linear filters, to the synthesis of radiation diagrams, and to interferometric measurements."

1781:

Džrbašyan, M. M.; Nersesyan, A. B. Criteria of expansibility of functions in Dirichlet series. Izv. Akad. Nauk Armyan. SSR. Ser. Fiz.-Mat. Nauk **11** (1958), no. 5, 85–106. (Russian. Armenian summary)

La fonction $F(\sigma)$ étant continue sur (σ_0, ∞), posons, pour tout $\alpha>0$:

$$\frac{d_\sigma^{-\alpha}F(\sigma)}{d_\sigma\sigma^{-\alpha}}=\frac{1}{\Gamma(\alpha)}\int_\sigma^\infty (e^{-\sigma}-e^{-u})^{\alpha-1}e^{-u}F(u)du$$

(dérivée de Riemann-Liouville d'ordre $-\alpha$, avec changement de variable). Soit $\mu_0=0$, (1) $0<\mu_{k+1}-\mu_k\leqq 1$, $\alpha_k=1-(\mu_k-\mu_{k-1})$, et posons: $L^{(\mu_0)}F(\sigma)=F(\sigma)$,

$$L^{(\mu_k)}F(\sigma)=-\frac{d_\sigma^{-\alpha_k}}{d_\sigma\sigma^{-\alpha_k}}\left\{e^\sigma\frac{d}{d\sigma}L^{(\mu_{k-1})}F(\sigma)\right\}\quad(k\geqq 1),$$

les fonctions ainsi introduites étant supposées continues sur (σ_0, ∞), $e^\sigma(d/d\sigma)L^{(\mu_k)}(\sigma)$ étant absolument intégrable sur (σ_0, ∞), la limite $L^{(\mu_k)}F(\infty)=\lim_{\sigma\to\infty}L^{(\mu_k)}F(\sigma)$ existant. On dit alors que $F\in L(\mu_n;\sigma_0)$. Voici le théorème (essentiel) de l'auteur: si $F\in L(\mu_n;\sigma_0)$ (avec $\lim\mu_n=\infty$), $|L^{(\mu_k)}F(\sigma)|\leqq Me^{-\sigma\mu_k}\Gamma(1+\mu_k)$ $(\sigma\in(\sigma_0,\infty))$, on a $F(\sigma)=\sum(L^{(\mu_k)}F(+\infty)/\Gamma(1+\mu_k)e^{-\mu_k\sigma}$ $(\sigma\in(\sigma_0,\infty))$; si $F(\sigma)=\sum a_k e^{-\mu_k\sigma}$ sur (σ_0,∞), les μ_k satisfaisant aussi à la condition $\limsup \log k/\mu_k<l<\infty$, on a pour $\sigma_1=\sigma_0+l$: $F(\sigma)\in L(\mu_n;\sigma_1)$, et $|L^{(\mu_k)}F(\sigma)|\leqq AB^{\mu_k}\Gamma(1+\mu_k)$, où $a_k=L^{(\mu_k)}F(+\infty)/(1+\mu_k)$. *S. Mandelbrojt* (Paris)

1782:

Voronovskaya, E. V. Extremal trigonometric polynomials and their applications. Dokl. Akad. Nauk SSSR **129** (1959), 12–15. (Russian)

An extremal trigonometric polynomial is one whose maximum absolute value is 1 and which has a sufficiently large number of nodes (points of maximum deviation from zero); it is denoted by C_n or S_n if it is a pure cosine or pure sine polynomial. The author reduces problems about trigonometric polynomials to problems about algebraic polynomials and so can use her previous results [same Dokl. **99** (1954), 5–8, 193–196; MR **17**, 842] to solve extremal problems for trigonometric polynomials. Extremal C_n with at least $n+1$ nodes on $(-\pi,\pi]$ are in one-to-one correspondence with extremal algebraic polynomials Q_n with more than $\frac{1}{2}n+1$ nodes via $Q_n[\frac{1}{2}(1+\cos\theta)]=C_n(\theta)$ on $[0,\pi]$. Every problem on trigonometric polynomials C_n deviating least from zero under linear constraints on the coefficients is reducible to a corresponding one on algebraic polynomials. For sine polynomials the situation is more complicated since the correspondence between an algebraic polynomial P_{n-1} and an S_n is via $S_n(\theta)=\varphi(\frac{1}{2}(1+\cos\theta))$ where $\varphi(x)=P_{n-1}(x)2\{x(1-x)\}^{1/2}$. In particular, the author solves the problem of finding the trigonometric polynomial deviating least from a given $\sum_{n+1}^n B_k\sin k\theta$.

R. P. Boas, Jr. (Evanston, Ill.)

1783:

Zuhovickiĭ, S. I.; Èskin, G. I. Some remarks on the best approximation of differential equations by polynomials. Dokl. Akad. Nauk SSSR **127** (1959), 1158–1160. (Russian)

On considère le problème de recherche des polynomes d'un degré fixé $u_m=\sum_{k=1}^m \xi_k\varphi_k$ $\{\varphi_k=(\varphi_{1k},\cdots,\varphi_{nk})\}$, $|\varphi_k|=\max_i|\varphi_{ik}|$, qui minimisent la déviation maximale entre les membres gauches et droites d'un système d'équations différentielles $Lu=f$ $(u=(u_1,\cdots,u_n);\ f=(f_1,\cdots,f_n))$, définies dans un domaine G et dont la solution satisfait à certaines conditions à la frontière $lu|_\Gamma=\psi$. Ce problème d'approximation tchébychévienne par un polynome d'une fonction continue dans un compact peut être réduit, à la suite du choisissement des réseaux assez denses sur G et Γ, à celui de la meilleure approximation d'un système d'équations algébriques linéaires non compatibles, auquel on peut appliquer, par exemple, l'algorithme proposé par Zuhovickiĭ [Mat. Sb. (N.S.) **33** (**75**) (1953), 327–342; MR **15**, 354], dont le programme a été élaboré par Zuhovickiĭ et V. B. Leonova [Luc'kii Derž. Ped. Inst. Nauk. Zap. Fiz.-Mat. Ser. **6** (1958), no. 3, 21–36; MR **21** #3087]. Parmi les exemples donnés, mentionnons ici les applications de la méthode aux problèmes à la frontière pour les équations harmoniques et bi-harmoniques. *D. Mangeron* (Iași)

1784:

Ahiezer, N. I. Orthogonal polynomials on several intervals. Dokl. Akad. Nauk SSSR **134** (1960), 9–12. (Russian); translated as Soviet Math. Dokl. **1** (1961), 989–992.

Classical investigations on the asymptotic form of orthogonal polynomials suppose that the weight function has "few" zeros. The author investigates the entirely different case where the weight function is zero except on a finite set (E) of nonoverlapping intervals $[-1,\alpha_1]$, $[\beta_1,\alpha_2]$, $\cdots$, $[\beta_p,1]$ and inside the intervals has one of the forms $S(x)\{-R(x)\}^{-1/2}/t(x)$, $\{-R(x)\}^{1/2}/\{S(x)t(x)\}$, where $t(x)$ is continuous and positive for $x\in E$ and $S(x)=\prod(x-\alpha_j)$, $R(x)=(x+1)(x-1)\prod(x-\alpha_j)(x-\beta_j)$. With the aid of a two-sheeted Riemann surface having branch points at the endpoints of the intervals E, the author constructs a rational function $p(z,\{R(z)\}^{1/2})$ such that

$$p(z,\{R(z)\}^{1/2})=T_n(z,P)-\{R(z)\}^{1/2}\{S(z)\}^{-1}U_{n-1}(z;P),$$

where T_n and U_{n-1} are orthogonal polynomials with the two weights indicated above, t now being a polynomial P of even degree that is positive on E. The polynomials with one weight can be expressed in terms of those with a different weight. Hence for arbitrary t the orthogonal polynomials can be expressed in terms of those associated with a simpler approximating weight and their asymptotic properties determined. *R. P. Boas, Jr.* (Evanston, Ill.)

1785:

Buchwalter, Henri. Saturation dans un espace normé. C. R. Acad. Sci. Paris **250** (1960), 651–653.

The author establishes a connexion between classes of saturation and multipliers. Notations: E = normed space with elements x and dual E'; $\sum_i \langle x, f_i\rangle e_i$ = Fourier series of x with certain biorthogonal systems (f_i), (e_i); Γ = summability method defined by constants $\gamma_\omega{}^i$ such that $\Gamma_\omega(x) \sim \sum_i \gamma_\omega{}^i \langle x, f_i\rangle e_i \in E$; $\rho(\omega) > 0$ a function with $\lim \rho(\omega) = 0$ $(\omega \to \infty)$ and $\lim (1-\gamma_\omega{}^i)/\rho(\omega) = \lambda_i \neq 0$ $(\omega \to \infty)$; F = closed subspace of E' in the strong topology with dual F' and $\|x\|_E \leq a\|x\|_{F'}$ for an $a > 0$ and all x; $\Sigma(\Gamma)$ = class of saturation of Γ; $X_\omega = [x - \Gamma_\omega(x)]/\rho(\omega)$ for all x; $\delta_\omega{}^i = (1-\gamma_\omega{}^i)/[\lambda_i \rho(\omega)]$; $Q \subset E'$ is the space of polynomials $q = \sum \langle e_i, q\rangle f_i$; $\Delta_\omega(q) = \sum_i \delta_\omega{}^i \langle e_i, q\rangle f_i$. Theorems: (1) $\|x - \Gamma_\omega(x)\| = o[\rho(\omega)]$ implies $x = 0$. (2) X_ω converges weakly in F' to $X \in F'$ defined by $\langle X, f_i\rangle = \lambda_i \langle x, f_i\rangle$ for every $x \in \Sigma(\Gamma)$. (3) If $\|\Delta_\omega(q)\|_F \leq A\|q\|_F$ (A independent of ω and q), then $\Sigma(\Gamma)$ is just the linear manifold of x such that $\sum_i \lambda_i \langle x, f_i\rangle e_i \sim X \in F'$.—Therewith classes of saturation for the methods of Riemann (R_r), Riesz and Nörlund are determined. Example: $\Sigma(R_r) = \{x \mid x^r \in L_p\}$ for $E = L_p$ $(1 < p < \infty)$, $\{x \mid x^r \in C'\}$ for $E = L$, $\{x \mid x^r \in L_\infty\}$ for $E = C$.
G. Goes (Evanston, Ill.)

FOURIER ANALYSIS

See also 1782, 1797.

1786:

Flett, T. M. On the strong summability of Fourier series. J. London Math. Soc. **33** (1958), 311–326.

The author proves the following theorem, together with some variants. Let $f(\theta)$ be periodic and integrable in $(-\pi, \pi)$, $\phi(t) = f(\theta+t) + f(\theta-t)$. Set

$$\Phi_\alpha(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-u)^{\alpha-1}\phi(u)du, \qquad \phi_\alpha(t) = \Gamma(\alpha+1)t^{-\alpha}\Phi_\alpha(t).$$

Let $1 < k \leq 2$, $\alpha \geq 1/k'$, and $\beta > \alpha + 1/k$. Suppose that, for some s, $\int_0^\pi |\phi_\alpha(t) - s|^k dt = O(x)$, and that, for some $\mu > 0$, $\phi_\mu(x) \to s$, as $x \to 0^+$; then the Fourier series of f is summable $[C, \beta]_{k'}$ to s at the point θ. This theorem is related to a recent result of Chow [J. London Math. Soc. **33** (1958), 161–170; MR **20** #1884]. *E. M. Stein* (Chicago, Ill.)

1787:

Efimov, A. V. Approximation of functions with given modulus of continuity by Fourier sums. Izv. Akad. Nauk SSSR. Ser. Mat. **23** (1959), 115–134. (Russian)

Let $f(x) \sim a_0/2 + \sum (a_k(f)\cos kx + b_k(f)\sin kx)$ be a continuous function, $S_n(f, x)$ the partial sums of this series, $r_n(f, x) = f(x) - S_n(f, x)$, $\omega_1(\delta, f) = \sup_{|h| \leq \delta} \max_x |f(x+h) - f(x)|$, and $\omega_2(\delta, f) = \sup \max |f(x+h) - 2f(x) + f(x-h)|$. Let $\omega(t) \geq 0$ be a fixed function, continuous at $t = 0$, satisfying $\omega(0) = 0$, $0 \leq \omega(t_2) - \omega(t_1) \leq \omega(t_2 - t_1)$ for $0 \leq t_1 \leq t_2$. Let $g(x)$ be a fixed continuous function and let $\omega_2(\delta) = \omega_2(\delta, g)$. Let us write: $f \in H_\omega{}^1$ if $\omega_1(\delta, f) \leq \omega(\delta)$; $f \in H_\omega{}^2$ if $\omega_2(\delta, f) \leq \omega_2(\delta)$ and $\omega_2(\lambda\delta) \leq (\lambda+1)\omega_2(\delta)$ for $\lambda > 0$; $C_i{}^{(n)}(\omega) = \sup |a_n(f)|$ for $f \in H_\omega{}^i$, $(i = 1, 2)$, and $\varphi \in \tilde{H}_\omega{}^2$ if $\varphi = f(x) + ax + b$, where $f \in H_\omega{}^2$ and a, b are constants.

The author proves the following results. 1. If $f \in \tilde{H}_\omega{}^2$, $f(0) = f(d) = 0$, then $f(x) = O(\ln(2d/x)\omega_2(x))$ for $x \in [0, d]$. 2. If $f \in H_\omega{}^2$ then $|r_n(f, x)| \leq (1/\pi)C_2{}^{(n)}(\omega)\ln n + O(\omega_2(1/n))$. 3. $\sup_{x, f \in H_\omega{}^1} |r_n(f, x)| = (1/\pi)C_1{}^{(n)}(\omega)\ln n + O(\omega_1(1/n))$. These theorems generalize known results of Zygmund and Nikolski. *M. Cotlar* (Buenos Aires)

1788:

Ganzburg, I. M.; Timan, A. F. Riemann sums for integrals of moduli of certain trigonometric polynomials. Uspehi Mat. Nauk **14** (1959), no. 3 (87), 123–128. (Russian)

Let

$$K_n(t) = \tfrac{1}{2}\lambda_0{}^{(n)} + \sum_{k=1}^{n} \lambda_k{}^{(n)} \cos kt,$$

where the $\lambda_k{}^{(n)}$ are real and either convex or concave, and $|\lambda_k{}^{(n)}| \leq A + B|\lambda_0{}^{(n)}|$. The Riemann sums

$$\sigma_n{}^r(x, \lambda) = 2\pi^{-1}r^{-1} \sum_{\nu=1}^{r} |K_n(x - t_\nu)|, \quad t_\nu = 2\nu\pi/r,$$

for the integrals $\int_0^{2\pi} |K_n(x-t)|dt$ play a role in the theory of trigonometric approximation using the summation method determined by $\|\lambda_k{}^{(n)}\|$. The authors determine their asymptotic behavior when $n \to \infty$ and r is an integral multiple of $2n+1$, namely

$$\sigma_n{}^r(x, \lambda) = \frac{2(2n+1)}{r\pi} \csc\frac{\pi(2n+1)}{2r} \times \left| \cos\left\{\frac{(2n+1)\pi}{r}\left(\frac{1}{2} + \left[\frac{rx}{2\pi}\right] - \frac{rx}{2\pi}\right)\right\} \sum_{k=0}^{n} \frac{\lambda_k{}^{(n)}}{n-k+1} \right| + O(A + B|\lambda_0{}^{(n)}|),$$

where the O-term is uniform in n, x and $r/(2n+1)$.
R. P. Boas, Jr. (Evanston, Ill.)

1789:

Watanabe, Hiroshi. On conjugate function of several variables. Mem. Fac. Sci. Kyusyu Univ. Ser. A **12** (1958), 180–190.

The paper contains errors which invalidate the results stated. The author has acknowledged this in a letter to the reviewer; he also indicates his intention of withdrawing the paper. *E. M. Stein* (Chicago, Ill.)

1790:

Rooney, P. G. On the representation of functions as Fourier transforms. Canad. J. Math. **11** (1959), 168–174.

The author defines the operators $\mathfrak{F}_{k,i}[F]$ by

$$\mathfrak{F}_{k,i}[F] = \frac{(-ik/t)^{k+1}}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} \frac{1}{(x - ik/t)^{k+1}} F(x)dx \qquad (k = 1, 2, \cdots).$$

The author proves that F is the Fourier transform of an $f \in L^p$ $(1<p<2)$, if and only if $F \in L^q$ $(1/p+1/q=1)$, and there exists a constant M so that

$$\int_{-\infty}^{+\infty} |\mathfrak{F}_{k,t}[F]|^p dt \leq M \quad (k = 1, 2, 3, \cdots).$$

The result is related to the Laplace transform by the identity

$$\mathfrak{F}_{k,t}[F] = (k/t)^{k+1}(k!)^{-1} \int_0^\infty e^{-ky/t} y^k f(y) dy, \quad \text{if} \quad t>0.$$

(There is a similar identity if $t<0$.)

The author similarly characterizes those F which are Fourier transforms of f such that $|x|^{1-2/q} f(x) \in L^q$ $(2 \leq q < \infty)$. *E. M. Stein* (Chicago, Ill.)

1791:

Eymard, Pierre. Applications laissant stable l'ensemble des fonctions presque-périodiques sur certains groupes discrets. C. R. Acad. Sci. Paris **249** (1959), 2459–2461.

$A(G)$ denotes the set of all almost periodic functions on the group G. In this note, G_1 is abelian (and written additively), but G_2 need not be. A map ϕ of G_2 into G_1 is called A-regular if $f \circ \phi \in A(G_2)$ whenever $f \in A(G_1)$, where $(f \circ \phi)(x) = f(\phi(x))$. Theorem 1: If G_2 is countable and has no proper subgroup of finite index, then ϕ is A-regular if and only if $\phi(x) = h + \sigma(x)$ $(x \in G_2)$, where σ is a homomorphism of G_2 into G_1 and h is a fixed element of G_1. Theorem 2: Suppose G_2 is finitely generated. Choose an invariant subgroup H of G_2, of finite index p; suppose $x_1, \cdots, x_p$ are representatives of the cosets of H, $\sigma_1, \cdots, \sigma_p$ are homomorphisms of H into G_1, and $h_1, \cdots, h_p$ are elements of G_1. Setting $\phi(yx_j) = h_j + \sigma_j(y)$ for $y \in H$, ϕ is an A-regular map of G_2 into G_1. Conversely, every A-regular map of G_2 into G_1 is obtained in this manner.

{Reviewer's remark: It seems that theorem 1 can be strengthened by insertion of the word "invariant" before "subgroup".} *W. Rudin* (Madison, Wis.)

1792:

Hua, Loo-keng. A convergence theorem in the space of continuous functions on a compact group. Sci. Record (N.S.) **2** (1958), 280–284.

Let $\{\varphi_{ij}^l(U)\}$ be a complete set of representation coefficients for the unitary group U_n, so that any continuous function $u(U)$ has a unique representation

$$u(U) \sim \sum_l \sum_{i,j} c_{ij}^l \varphi_{ij}^l(U).$$

From a previous integral representation the author derives a certain system of functions $\Phi_{ij}^l(r, U)$, $0<r<1$, which corresponds to the system $\{r^{|n|}e^{in\theta}\}$ in the case of the exponentials $\{e^{in\theta}\}$, such that the expansion

$$u(r, U) = \sum_l \sum_{i,j} c_{ij}^l \Phi_{ij}^l(r, U)$$

is uniformly convergent for each r, and $u(r, U) \to u(U)$ as $r \to 1$, uniformly in U. *S. Bochner* (Princeton, N.J.)

INTEGRAL TRANSFORMS AND OPERATIONAL CALCULUS

See also 1642, 1811.

1793:

Widom, Harold. A theorem on translation kernels in n dimensions. Trans. Amer. Math. Soc. **94** (1960), 170–180.

The following theorem, a generalization of a result of M. Kac [Duke Math. J. **21** (1954), 501–509; MR **16**, 31] to the space E_n of n real variables, is established. Let $x \in E_n$, $\rho(x)$ be real and even, $\int_{E_n} (1+|x|)|\rho(x)|dx < \infty$, and assume $F(y) = \int_{E_n} e^{ix \cdot y} \rho(x) dx$ belongs to $L(E_n)$. Let K be a compact subset of E_n which is the closure of its interior, and denote by $\lambda_1(a), \lambda_2(a), \cdots$ the eigenvalues of the integral equation $\int_{aK} \rho(x-y)\phi(y)dy = \lambda\phi(x)$. Then for sufficiently small λ, we have, as $a \to \infty$,

$$\sum_{j=1}^{\infty} \log[1 - \lambda \cdot \lambda_j(a)] =$$

$$a^n V(K)(2\pi)^{-n} \int_{E_n} \log[1-\lambda F(y)]dy + \tfrac{1}{2}a^{n-1} \int_{\partial K} d\sigma \times \int_{x \cdot \xi \geq 0} x \cdot \xi \left\{ (2\pi)^{-n} \int_{E_n} \log[1-\lambda F(y)] e^{ix \cdot y} dy \right\}^2 dx + o(a^{n-1})$$

where $V(K)$ denotes the volume of K, ∂K the boundary of K, $d\sigma$ the surface element on ∂K, and ξ the unit outer normal at a point of ∂K.

The proof is based upon a group of lemmas concerning the volume of certain subsets of K and of its translates aK, which may be of interest to persons interested in other problems which involve integration in E_n. *D. L. Bernstein* (Towson, Md.)

1794:

Hirschman, I. I., Jr. Hankel transforms and variation diminishing kernels. Bull. Amer. Math. Soc. **66** (1960), 40–42.

The author states, without proof, a theorem on variation diminishing # kernels. $J_\nu(x)$ is a Bessel function of order ν, γ is fixed, $\gamma \geq 0$.

$$T(x) = 2^{\gamma-1/2}\Gamma(\gamma+1/2)x^{1/2-\gamma}J_{\gamma-1/2}(x),$$
$$\mu(x) = x^{2\gamma+1}/2^{\gamma+1/2}\Gamma(\gamma+3/2),$$
$$\hat{f}(t) = \int_0^\infty T(xt)f(x)d\mu(x).$$

If a triangle Δ can be formed with sides x, y, z then $A(x, y, z)$ is equal to the area of Δ and if no such triangle exists then $A(x, y, z) = 0$.

$$\Gamma(\gamma)\pi^{1/2}D(x, y, z) = 2^{3\gamma-5/2}\Gamma(\gamma+1/2)^2(xyz)^{-2\gamma+1}A(x, y, z)^{2\gamma-2},$$

$$f \# g(x) = \int_0^\infty \int_0^\infty f(y)g(z)D(x, y, z)d\mu(y)d\mu(z).$$

On $(0, \infty)$ $H(x)$ is measurable and $H(x) \geq 0$; $\int_0^\infty H(x)d\mu(x) = 1$; $\psi(x)$ is continuous on $(0, \infty)$, and $V[\psi]$ denotes the number of changes of sign of $\psi(x)$ as x varies from 0 to ∞.

Definition: $H(x)$ is a variation diminishing # kernel if $V[H \# \psi] \leq V[\psi]$. The theorem stated by the author is the

following. $H(x)$ is a variation diminishing # kernel if and only if $\hat{H}(t)$ is of the form

$$\left[e^{ct^2}\prod_k\left(1+\frac{t^2}{a_k^2}\right)\right]^{-1},$$

where c is non-negative, a_k is real and $\sum_k 1/a_k^2$ is finite.

C. Fox (Montreal)

1795:

Saltz, Daniel. An inversion theorem for Laplace-Stieltjes transforms. Pacific J. Math. **10** (1960), 309–312.

Generalizing an inversion formula of Doetsch for Laplace transforms, the author proves the following result: If α is a normalized function of bounded variation on $[0, R]$ for every $R>0$, and if $f(s)=\int_0^\infty e^{-st}d\alpha(t)$ converges for some $s>0$ then, for $t>0$,

$$\lim_{s\to\infty}\sum_{n=1}^{\infty}[(-1)^{n+1}/n!]f(ns)e^{nst} = \alpha(t)-\alpha(0)-[\alpha(t^+)-\alpha(t^-)](e^{-1}-\tfrac{1}{2}).$$

R. R. Goldberg (Evanston, Ill.)

1796:

Murav'ev, P. A. On finding the original function when the transformed function is a proper rational fraction. Vyčisl. Mat. **5** (1959), 141–148.

A computational method for obtaining explicitly the coefficients of the residue polynomial factors of the exponentials in the inverse Laplace transform of rational functions is presented. It is assumed that the latter are complex, that the poles have arbitrary multiplicity and negative real parts and are explicitly known, and that the denominator degree exceeds that of the numerator. The expressions for the residue coefficients are derived for the case of constant numerator. The method is straightforward, and the expressions involve multiple convolutory sums over the denominator coefficients and the pole location differences; they seem suitable for computer programming. Specialization to the case of real coefficients follows, with consequent simplification. The case of non-constant denominator simply involves the previously derived expressions and their derivatives up to the requisite order. Numerical examples are given.

H. G. Baerwald (Albuquerque, N.M.)

1797:

Goldberg, R. R. An integral transform related to the Hilbert transform. J. London Math. Soc. **35** (1960), 200–204.

The author discusses the transform

$$\text{(i)}.\qquad f^*(x)=\frac{1}{\pi}\int_0^\infty\frac{f(x+t)-f(x-t)}{t}\cdot\frac{\sin t}{t}\,dt,$$

introduced by Boas [Trans. Amer. Math. Soc. **40** (1936), 287–308].

Writing $g(x)$ for the Hilbert transform of $f(x)$, $\delta(t)$ for $(2/\pi)^{1/2}(1-\cos t)t^{-2}$, and using the symbol "$\circ$" to denote the Fourier convolution, it is proved that

$$f^* = g-g\circ\delta,$$
$$f^{**} = -f+2\delta\circ f-\delta\circ\delta\circ f.$$

From these results an inversion formula for (i) is found. The author is then able to prove easily two theorems from the reference quoted above. These theorems deal with the vanishing of Fourier transforms in the interval $(-1, +1)$.

J. L. Griffith (Kensington)

INTEGRAL AND INTEGRODIFFERENTIAL EQUATIONS

1798:

Coz, Marcel. Résolution d'une équation intégrale liée à la représentation des pyramides et cônes asymétriques. C. R. Acad. Sci. Paris **249** (1959), 1175–1177.

1799:

Lavrent'ev, M. M. Integral equations of the first kind. Dokl. Akad. Nauk SSSR **127** (1959), 31–33. (Russian)

Consider the integral equation of the first kind (1) $A\phi=f$, where A is a completely continuous operator. By restricting himself to the class of solutions ϕ of the form $\phi=B\psi$, $\|\psi\|\leq 1$, where B is completely continuous and ψ is an element of Hilbert space, and by further assuming that if u satisfies $\|u\|\leq 1$, $\|ABu\|\leq\varepsilon$ then $\|Bu\|\leq\delta(\varepsilon)$ (L_2 norms), the author describes an "effective method" of solution of (1). This means that an algorithm is determined which permits the calculation of ϕ with a guaranteed degree of accuracy, whenever f_α approximates f up to some preassigned α (i.e., $\|f_\alpha-f\|\leq\alpha$). Such restrictions are needed, because in general (1) does not constitute a properly posed problem. Equations of the form (1) arise, e.g., in the Cauchy problem for Laplace's equation previously treated by the author [Izv. Akad. Nauk SSSR. Ser. Mat. **20** (1956), 819–842; MR **19**, 426]. A specific such Cauchy problem, including the determination of operators A and B, and a stability estimate, are given.

J. A. Nohel (Atlanta, Ga.)

1800:

Genčev, T. Über die unzerlegbaren Vektoren gewisser Kegel. Bŭlgar. Akad. Nauk Izv. Mat. Inst. **3**, no. 1, 69–88 (1958). (Bulgarian. Russian and German summaries)

Sei K ein abstrakter normierter Kegel und $P(f)$ eine Norm in ihm. Nach Tagamlicki [Ann. Univ. Sofia Fac. Sci. Phys. Math. Livre 1, **48** (1953/54), 69–85; MR **17**, 767] ein Element $f\in K$ wird bezüglich der Norm $P(f)$ unzerlegbar genannt im Falle dass $f\neq 0$, und $f=g+h$ und $P(f)=P(g)+P(h)$ $(g\in K, h\in K)$ dann und nur dann gelten, wenn $g=\lambda f$ und $h=\mu f$ für $\lambda, \mu\geq 0$.

Der Autor betrachtet hier speziell den Kegel L der symmetrischen stetigen Kerne $K(x,t)$, die im Quadrat Δ: $a\leq x\leq b, a\leq t\leq b$ positive (im Sinne $\int_\Delta K(x,t)p(x)p(t)dxdt\geq 0$ für jede im Quadrat integrierbare $p(x)$) sind, und beweist: Mit der Norm $P(K)=\int_a^b K(x,x)dx$ ist der Kern $K\in L$ unzerlegbar dann und nur dann, wenn es eine in $[a,b]$ stetige Funktion $u(x)$ gibt, für die $K(x,t)=u(x)\cdot u(t)$ gilt.

Ähnlicher Satz wird für die Kerne, die sich durch schwach konvergente Folgen $\{K_\nu\}$ von Kernen aus L approximieren lassen, aufgestellt und einige weitere Sätze über Zerlegung solcher Kerne bewiesen.

V. Vučković (Belgrade)

1801:

Hilzman, John. Error bounds for an approximate solution to the Volterra integral equation. Pacific J. Math. **10** (1960), 203–207.

In a paper written in 1945, Michal [Proc. Nat. Acad. Sci. U.S.A. **31** (1945), 252–258; MR **7**, 304] considered the resolvent kernel k and the solution $y(x)$ as functionals of the kernel K of the Volterra integral equation $y(x)+\int_a^x K(x,t)y(t)dt=f(x)$. In this paper the author utilizes the

suggestions given by Michal to obtain approximations to the solution with error bounds for any order of approximation by the use of the Fréchet differentials of k and y.

I. A. Barnett (Cincinnati, Ohio)

1802:

Fen'e, Ištvan [Fenyő, István]. On the theory of Volterra integral operators. Dokl. Akad. Nauk SSSR **125** (1959), 51–54. (Russian)

The author describes two linear operators k and l in the Hilbert space $L^2[0, a]$ as being linearly equivalent if there is a bounded linear operator V with a bounded inverse such that $k = VlV^{-1}$.

He gives an elementary proof, without using the theory of characteristic matrix functions, of the following result. Let $K(x, y)$ be the kernel of a Volterra integral operator k on $L^2[0, a]$, and suppose that $K(x, x) \neq 0$ in $[0, a]$ and that $\partial^2 K/\partial x \partial y$ exists and is bounded. Then the operator k is linearly equivalent to the integral operator I defined by $(If)(x) = \int_0^x f(t)dt$.

Using this result, together with some elementary formulae from the theory of integration of fractional order, he obtains a new proof of a theorem of L. A. Sahnovič [Izv. Akad. Nauk SSSR Ser. Mat. **22** (1958), 299–308; MR **20** #5409] on the existence of nth roots of Volterra integral operators of the closed cycle; he also generalises this result to Volterra operators satisfying the conditions of his main theorem and to certain others.

F. Smithies (Cambridge, England)

1803:

Bharucha-Reid, A. T. On random solutions of Fredholm integral equations. Bull. Amer. Math. Soc. **66** (1960), 104–109.

This paper is a continuation of the author's work on random Fredholm integral equations. First the notion of a random boundary value problem is given together with a precise definition of the notion of a random solution of such an equation. Then the theory of random Fredholm integral equations on Orlicz spaces is discussed. Some results of the classical theory of Fredholm integral equations are generalized to this case.

W. A. J. Luxemburg (Pasadena, Calif.)

FUNCTIONAL ANALYSIS

See also 1669.

1804:

Wilansky, Albert; Zeller, Karl. FH-spaces and intersections of FK-spaces. Michigan Math. J. **6** (1959), 349–357.

An FH space L is an F-space (locally convex space of type (F)) continuously embedded in a Hausdorff space H. With $H = (s)$ one obtains the FK spaces [Zeller, Math. Z. **53** (1951), 463–487; MR **12**, 604]. The theory of the latter is extended to the former. Returning to FK spaces the authors "fill in the remaining gap in the theory of intersections in summability", defining a new property which lies between perfectness and the AK property [Zeller, Math. Z. **55** (1951), 57–70; MR **13**, 934].

R. Arens (Jerusalem)

1805:

Brainerd, Barron. On the embedding of vector lattices in F-rings. Trans. Amer. Math. Soc. **93** (1959), 132–144.

It is shown that every σ-complete vector lattice L with a weak order unit can be embedded in a σ-complete lattice-ordered ring R with an identity element that is a weak order unit. (The former is called an F-space; the latter an F-ring.) The embedding is used to investigate the structure of F-spaces. The main results of the paper follow.

Suppose that the Boolean algebra of idempotents of R admits a countably additive (normal) measure, and let $U = \{f \in R : \mu(\bigvee_{k=1}^{\infty} (1 \wedge |f|)) = 0\}$. Then U is an ideal of R such that the quotient ring R/U is a complete F-ring that is regular (in the sense of von Neumann). This is used to show that if μ is a nonnegative order-continuous linear function on L and $V = \{f \in L : \mu(|f|) = 0\}$, then L/V is a complete F-space.

It is also shown that every non-trivial bounded linear functional on a regular F-ring R is a finite linear combination of ring homomorphisms of R onto the real field.

M. Henriksen (Detroit, Michigan)

1806:

Brainerd, Barron. On the embedding of a vector lattice in a vector lattice with weak unit. Nederl. Akad. Wetensch. Proc. Ser. A **63** = Indag. Math. **22** (1960), 25–31.

The author shows that every vector lattice V can be embedded in a vector lattice L with weak unit so that every positive element of L is a supremum of elements in V. He constructs such an L making use of a certain extension of V given by the reviewer [J. Fac. Sci. Hokkaido Univ. Ser. I **12** (1953), 111–156; MR **15**, 137].

{Reviewer's remark: The definition given by the author for the projector of an element of a vector lattice is wrong and should be corrected to that used in the paper cited above.}

I. G. Amemiya (Tokyo)

1807:

Warner, Seth. Bornological structures. Illinois J. Math. **4** (1960), 231–245.

The present paper offers among others a framework with which various "bornological concepts" can be unified. A structure on a real vector space E is a filter base $\mathscr{V}$ of convex equilibrated subsets of E such that $\{E\} \in \mathscr{V}$ and such that $V \in \mathscr{V}$ implies $\lambda V \in \mathscr{V}$ for each real number λ, and a structured space is a pair $(E, \mathscr{V})$ such that $\mathscr{V}$ is a structure on E. A locally convex topology $\mathscr{T}$ on E is compatible with the structure $\mathscr{V}$ if there is a subfamily of $\mathscr{V}$ which is a local base for the $\mathscr{T}$-neighborhoods of zero. A structured locally convex space is a triple $(E, \mathscr{V}, \mathscr{T})$ where $\mathscr{V}$ is a structure on E and the topology $\mathscr{T}$ is compatible with $\mathscr{V}$. In addition let $\mathscr{B}$ be a family of subsets of E; then a subset V of E is a $\mathscr{B}$-bornivore set if $V \in \mathscr{V}$, V is absorbing, and V absorbs each B in $\mathscr{B}$. $(E, \mathscr{V}, \mathscr{T})$ is called a $\mathscr{B}$-bornological structure if each B of $\mathscr{B}$ is $\mathscr{T}$-bounded and each $\mathscr{B}$-bornivore subset of E is a $\mathscr{T}$-neighborhood of zero. We cite one class of specializations: if E is a partially ordered vector space then the order structure consists of all non-empty equilibrated convex and order-convex subsets of E (A is order-convex if $a \leq x \leq b$ and $a, b \in A$ imply $x \in A$). Thus by varying $\mathscr{B}$ one arrives at o-bornological or P-o-bornological spaces, introduced by J. Kist [Duke Math. J. **25** (1958), 569–582; MR **20** #4762],

and order-bound topologies due to the reviewer [Mem. Amer. Math. Soc. no. 24 (1957); MR **20** #1193] and to H. Schaefer independently [Math. Ann. **135** (1958), 115–141; MR **21** #5134]. If $\{(E_\alpha, \mathscr{V}_\alpha, \mathscr{T}_\alpha): \alpha \in A\}$ is a family of structured locally convex spaces, then the product structured locally convex space is defined, and the main theorem gives a relation between the non-existence of Ulam measure on A and the inheritance of bornological properties by the product. The standard theorem concerning the product of bornological spaces is one of many consequences of the main theorem. The paper concludes with the section where the author discusses relationships among various bornological properties of order structured locally convex spaces. *I. Namioka* (Ithaca, N.Y.)

1808:

Kelley, J. L. Hypercomplete linear topological spaces. Bull. Soc. Math. Belg. **10** (1958), 2–3.

Abstract of Michigan Math. J. **5** (1958), 235–246 [MR **21** #2173].

1809:

Bae, Mi-Soo. On the regularity of the locally uniformly convex Banach space. Kyungpook Math. J. **2** (1959), 61–64.

It is proved that every locally uniformly convex (luc) Banach space which satisfies an additional assumption ("condition (A)") is reflexive. {The author overlooks the fact that condition (A) and luc imply that the space is uniformly convex.} *R. R. Phelps* (Berkeley, Calif.)

1810:

Zaidman, Samuel. Sur le problème vectoriel des moments. C. R. Acad. Sci. Paris **250** (1960), 436–438.

Let X be a Banach space, X' its dual. A function $\alpha(t)$ from $[0, 1]$ to X is said to belong to $V_f(X)$ if for every $x' \in X$, $\langle \alpha(t), x' \rangle$ is of bounded variation. D. C. Burgess has shown [Proc. London Math. Soc. (3) **4** (1954), 107–128; MR **15**, 631] that if X is reflexive and $\{\mu_n\}$ is a sequence in X, then $\mu_n = \int_0^1 t^n d\alpha(t)$ for $\alpha(t) \in V_f(X)$ if and only if for every $x' \in X'$ there exists an L so that

$$\sum_{m=0}^{k} \left| \binom{k}{m} (-1)^{k-m} \Delta^{k-m} \langle \mu_m, x' \rangle \right| < L.$$

The author announces that this result is true under the weaker condition that X is weakly sequentially complete. He also points out that the theorem is no longer true if one removes all restrictions on X.

Designate by $V_G(X)$ [$V_S(X)$] the Gel'fand [Sirvint] classes of functions of bounded variation, i.e.,

$$\sum_{0}^{n} \varepsilon_i (\alpha(t_{i+1}) - \alpha(t_i)), \quad \varepsilon_i = \pm 1,$$

are relatively compact [weakly compact] in X. Then if X is any Banach space, a sequence $\{\mu_n\}$ in X has an integral representation by a function in V_G [V_S] if and only if the set of elements

$$\left\{ \sum_{m=0}^{n} \varepsilon_{n,m} \binom{n}{m} (-1)^{n-m} \Delta^{n-m} \mu_m \right\}, \quad \varepsilon_{n,m} = \pm 1,$$

is relatively compact [weakly compact] in X.

Further theorems are announced which give conditions that $d\alpha$ be absolutely continuous. These theorems present many analogies with the previous work of the author [e.g., Ann. of Math. (2) **68** (1958), 260–277; MR **21** #803] on the representation of functions by Laplace-Stieltjes integrals. *A. Devinatz* (Princeton, N.J.)

1811:

Kipriyanov, I. A. Fractional derivative and imbedding theorems. Dokl. Akad. Nauk SSSR **126** (1959), 1187–1190. (Russian)

Suppose $f(Q)$ is a summable function defined in a convex region Ω of n-dimensional euclidean space. Let P be a fixed point of Ω and $Q(r, e)$ an arbitrary point of this region, where e is the unit vector in the direction from P to Q, and r is the distance between P and Q. If there exists a summable function $f^{(\alpha)}(P, Q)$ such that

$$\int_0^r f^{(\alpha)}(P, P+et) t^{n-1} dt = \frac{1}{\Gamma(1-\alpha)} \int_0^r (r-t)^{-\alpha} [f(P+et) - f(P)] t^{n-1} dt,$$

it is called the fractional derivative of f of degree α at the point Q in the direction e. If the above equation is true for all P, Q in Ω, then $f^{(\alpha)}$ is called the fractional derivative of f of degree α.

If $f^{(\alpha)}$ is bounded in (P, Q), then $f \in \mathrm{Lip}\, \alpha$. If $f^{(\alpha)} \in L_p$ $(p > 1)$ and $\alpha > 1/p$, then $f \in \mathrm{Lip}\,(\alpha - 1/p, p)$. Conversely, if $0 < \alpha < \beta < 1$ and $f \in \mathrm{Lip}\, \beta$, then $f^{(\alpha)}$ exists and is continuous in (P, Q). If $f \in \mathrm{Lip}\,(\beta, p)$ $(p > 1)$ and $0 < \alpha < \min(\beta, n/p)$, then $f^{(\alpha)}$ exists and belongs to L_p.

If f is summable and $f^{(\alpha)}$ exists, set

$$\|f\|_{W_p^{(\alpha)}} = ||\Omega|^{-1} \int_\Omega f(Q) dQ| + \left(\int_\Omega \operatorname*{ess\,sup}_{P \in \Omega} |f^{(\alpha)}(P, Q)|^p dQ \right)^{1/p},$$

provided the last term is finite. A number of results are given relating this norm to various other norms. A representative example is the following: If $f \in W_p^{(\alpha)}$ and $\alpha p > n$, then $f \in C(\Omega)$ and $\|f\|_C \leq A \|f\|_{W_p^{(\alpha)}}$.

A. Devinatz (Princeton, N.J.)

1812:

Solonnikov, V. A. Certain properties of $\mathfrak{B}_p^l$ spaces of fractional order. Dokl. Akad. Nauk SSSR **134** (1960), 282–285 (Russian); translated as Soviet Math. Dokl. **1** (1961), 1071–1074.

A $\mathfrak{B}_p^l(E_n)$ space is defined as the closure of all schlicht functions on E_n, with norm given by

$$\|f\|_{\mathfrak{B}_p^l} = \sum_{i,j=1}^{n} \left(\int_0^\infty \frac{dh}{h^{1+p\lambda}} \int_{E_n} |D_i^l f(x_1, \cdots, x_j + h, \cdots, x_n) + D_i^l f(x_1, \cdots, x_j - h, \cdots, x_n) - 2 D_i^l f(x_1, \cdots, x_n)|^p dx \right)^{1/p},$$

where $l = \bar{l} + \lambda$, $0 < \lambda \leq 1$, $D_i^l = \partial^l / \partial x_i^l$.

The author investigates the inclusion of one such space in another, under various conditions on p and l.

S. Hoffman (Hartford, Conn.)

1813:

Roumieu, Charles. Nouvelles classes de distributions généralisées. C. R. Acad. Sci. Paris **248** (1959), 346–348.

The author states results concerning the class $L^2(\{M_p\}, \{N_p\})$ of functions satisfying

$$\int_{-\infty}^{+\infty} |f^{(p)}(x)|^2 e^{2q|x|}dx \leqq A h^{2p} e^{2rq} \cdot M_p^2 \cdot N_p^2 \qquad (p = 0, 1, 2, \cdots)$$

and its dual space $L^{2'}(\{M_p\}, \{N_p\})$ of distributions. A particular result is as follows. If $\varphi \in L^{2'}(\{M_p\}, \{N_p\})$ and the constants $\{M_p\}$ and $\{N_p\}$ satisfy appropriate restrictions, then

$$\varphi = \lim_{\substack{\eta>0 \\ \eta\to 0}} [\varphi_1(x-i\eta) - \varphi_2(x+i\eta)],$$

where φ_2 and φ_1 are holomorphic in the upper half-plane and lower half-plane respectively, and φ_2 and φ_1 satisfy growth conditions of the form

$$\int_{-\infty}^{+\infty} |\varphi(x+iy)|^2 \exp[-2N(|x|-\gamma)]dx \leqq A.$$

E. M. Stein (Chicago, Ill.)

1814:

Matthews, G. Transposes of associative g-rings. Nederl. Akad. Wetensch. Proc. Ser. A **62** = Indag. Math. **21** (1959), 508–511.

An associative g-ring is a set of infinite matrices which is closed under finite sum and product, and which obeys the associative law for multiplication. It differs from a Köthe-Toeplitz ring [see the reviewer's *Linear operators*, Macmillan, London, 1953; MR **15**, 719; p. 311] in that absolute convergence of the series $\sum_{j=1}^{\infty} a_{i,j}b_{j,k}$ occurring in the product of two matrices is not required. The set of matrices which transform a sequence space α into itself is denoted by $\mathscr{M}(\alpha)$. Conditions have been given for $\mathscr{M}(\alpha)$ to be an associative g-ring [G. Matthews, Nederl. Akad. Wetensch. Proc. Ser. A. **61** (1958), 298–306; MR **21** #813b], using the g-dual of α, denoted by $\alpha^\dagger$. In particular, if $\alpha^{\dagger\dagger} = \alpha$, α is called g-perfect. The transpose of a matrix A is, as usual, denoted by A', and the transpose of $\mathscr{M}(\alpha)$ (i.e., the set of matrices which are transposes of those in $\mathscr{M}(\alpha)$) is denoted by $\mathscr{M}'(\alpha)$. For meanings of ϕ, and of K, T, and α matrices, see the reviewer's *Infinite matrices and sequence spaces* [Macmillan, London, 1950; MR **12**, 694], pp. 63, 65, 94 respectively. The following results are established. Theorem 2.I: If (i) $\alpha \geqq \phi$, and (ii) $\mathscr{M}(\alpha)$, $\mathscr{M}(\alpha^\dagger)$ are associative g-rings, then $\mathscr{M}'(\alpha) \leqq \mathscr{M}(\alpha^\dagger)$, equality holding if and only if α is g-perfect. Theorem 3.I: Let $\mathscr{T}$ be a Köthe-Toeplitz ring and $\mathscr{S}$ a g-ring of infinite matrices. Then $\mathscr{T} \cap \mathscr{S}'$ is a Köthe-Toeplitz ring. Corollary: $\mathscr{S} \cap \mathscr{T}'$ is also a Köthe-Toeplitz ring. A particular case of this last result is the following, due to P. Vermes [Colloque sur la théorie des suites (Bruxelles, 1957), pp. 60–86, Centre Belge des Recherches Math., 1958; MR **20** #7163]. Theorem 3.II: (i) If A, B are α-T matrices, then so is AB; (ii) if E, F are T-α matrices, so is EF. (α-T matrices means α-matrices whose transposes are T-matrices.) The associative g-ring of δ-matrices A is defined by (a) $\sum_{n=1}^{\infty} a_{n,k}$ is convergent for all k,

$$\text{(b)} \qquad \sum_{k=1}^{\infty} \left| \sum_{i=1}^{n} (a_{i,k} - a_{i,k+1}) \right| \leqq M$$

for all n. K-matrices which transform all sequences $\{s_k\}$ such that $\sum |s_k - s_{k+1}| < \infty$ into another such sequence are called absolute-K, or $|K|$, matrices. δ-matrices which also transform every sequence $\{u_i\}$ such that $\sum |u_i|$ converges into another such sequence are called absolute-δ, or $|\delta|$, matrices. Theorem 3.III: (i) A K-matrix is K-δ if and only if it is a $|K|$-matrix; (ii) $|K|$-matrices form a Köthe-Toeplitz ring $\mathscr{T}$.

Theorem 3.IV: (i) A δ-matrix is δ-K if and only if it is a $|\delta|$-matrix; (ii) $|\delta|$-matrices form the Köthe-Toeplitz ring $\mathscr{T}'$, where $\mathscr{T}$ is defined in Theorem 3.III.

These last two theorems are analogues of conditions given by P. Vermes, in his paper cited above (theorems 3.I, 3.II) for (i) a T-matrix to be T-α, (ii) an α-matrix to be α-T. If λ is the space of sequences $\{x_i\}$ such that $\sum x_i$ has bounded partial sums, a matrix belonging to $\mathscr{M}(\lambda)$ is called an ε-matrix; if such a matrix also preserves absolute convergence of $\sum x_i$, it is called an absolute-ε, or $|\varepsilon|$, matrix.

Theorem 3.V: (i) An ε-matrix is ε-K_r if and only if it is an $|\varepsilon|$-matrix; (ii) $|\varepsilon|$-matrices form a Köthe-Toeplitz ring.

R. G. Cooke (London)

1815:

Kendall, David G. On infinite doubly-stochastic matrices and Birkhoff's problem 111. J. London Math. Soc. **35** (1960), 81–84.

The author introduces a new topology on the space of row-and-column-summable infinite real matrices with respect to which the set of doubly stochastic matrices is precisely the closed convex hull of the set of permutation matrices. [Cf. Rattray and Peck, Trans. Roy. Soc. Canada. Sect. III (3) **49** (1955), 55–57; MR **17**, 778.] He also proves topologically that the permutation matrices are the only extreme points of the set of doubly stochastic matrices, and raises the problem of an algebraic proof of this result.

J. Isbell (Seattle, Wash.)

1816:

Altman, M. On the extension of linear transformations in Banach spaces. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **6** (1958), 241–248. (Russian summary, unbound insert)

Several extension theorems for particular classes of linear maps of a Banach X space into itself or its dual X^* are established. A linear map A of X into itself is called a Fredholm endomorphism if there exists a constant c such that $|\text{trace } AK| \leqq c \cdot \|K\|$ for all linear maps K of X onto finite-dimensional subspaces of X. (For a map K with finite-dimensional range, $K(x) = \sum_{i=1}^{n} \varphi_i(x) \cdot x_i$ for appropriate $x_i \in X$ and $\varphi_i \in X^*$, and trace $K = \sum_{i=1}^{n} \varphi_i(x_i)$.) The following Theorem 1 is typical: Every Fredholm endomorphism A of a reflexive Banach space X can be extended to an arbitrary reflexive Banach space $Y \supset X$; the norm of the extended endomorphism does not exceed c. From his extension theorems the author derives a result on extensions of bilinear functionals due to G. Marinescu [Bull. Math. Soc. Roumaine. Sci. **47** (1946), 202–209; MR **9**, 448], as well as results on the existence of linear maps with preassigned values.

B. Grünbaum (Seattle, Wash.)

1817:

Bharucha-Reid, A. T. On random operator equations in Banach spaces. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **7** (1959), 561–564. (Russian summary, unbound insert)

The classical formula $R_\lambda = -\sum_{n=1}^{\infty} \lambda^{-n} T^{n-1}$ for the resolvent $R_\lambda = (T-\lambda)^{-1}$ of a bounded linear operator T on a Banach space is extended to the case in which T depends upon a stochastic parameter.

H. P. McKean, Jr. (Cambridge, Mass.)

1818:

Dolph, C. L.; Penzlin, F. On the theory of a class of non-self-adjoint operators and its applications to quantum scattering theory. Ann. Acad. Sci. Fenn. Ser. A. I. no. 263 (1959), 36 pp.

The theoretical half of this paper largely concerns operators of the form $R+iS$, where R and S are self-adjoint operators in a Hilbert space and S has absolutely convergent trace. Results are given relating the spectrum of A to those of R and S. More generally, if R and S are hermitian, if $S \geqq 0$, and if $(A-\lambda I)$ has range equal to the entire space, where $A = R+iS$, for $\operatorname{Im}\lambda > 0$, then $(A-\lambda I)^{-1}$ exists as a bounded operator in the half-plane $\operatorname{Im}\lambda > 0$ and has a weak representation as a Stieltjes transform there. {The statement and proof of this result need to be modified in an obvious way to take into account the non-hermitian character of the relevant spectral function.} As the authors note, a closely related result has been announced by Glazman [Dokl. Akad. Nauk SSSR **115** (1957), 214–216; MR **19**, 969].

Half the paper is devoted to applications to quantum scattering and consists largely of an extended correlated exposition of recent physical and mathematical developments on the subject, including in particular the work of Dolph and Ritt [Math. Z. **65** (1956), 309–326; MR **18**, 360]. A substantial and rather up-to-date bibliography (as of 1957) is included.

I. E. Segal (Chicago, Ill.)

1819:

Gribanov, Yu. I. Necessary and sufficient conditions for the existence of a unique solution to an infinite system of linear equations. Dokl. Akad. Nauk SSSR **129** (1959), 1211–1213. (Russian)

Let E be the infinite identity matrix; P_n the matrix obtained from E by replacing the first n (diagonal) 1's by 0's; $E_n = E - P_n$; and let $A = ((a_{mn}))$ $(m, n = 1, 2, \cdots)$ be a continuous matrix operator (with norm $\|A\|$) in the Banach space l of all infinite number sequences $X = \{x_n\}$. Define infinite matrices $A_n = E_n A E_n$, $A_{n\infty} = E_n A$, $A_{\infty n} = A E_n$. Operator A is called (respectively) ω-, α-, β-completely continuous (hereafter c.c.) according as $\lim \|A - A_n\| = 0$, $\lim \|A - A_{n\infty}\| = 0$, $\lim \|A - A_{\infty n}\| = 0$ for $n \to \infty$. Let Δ_n be the nth order determinant $((\delta_{ij} - a_{ij}))$ where δ_{ij} = Kronecker delta.

The infinite system of linear equations (1) $X - AX = H$ is called an ω-, α-, β-c.c. system if $H \in l$ and A is correspondingly ω-, α-, β-c.c. in space l; and (1) is said to be uniquely solvable in l if it has a unique solution $X = \{x_n\} \in l$ for each $H = \{h_n\} \in l$. Let B_n denote A_n, $A_{n\infty}$, $A_{\infty n}$ according as A is (resp.) ω-, α-, β-c.c. Seven theorems are stated, of which two give conditions for unique solvability, and the others give limit expressions for the unique solution together with estimates of the closeness of the approximating sequences. Sample results: Theorems 1 and 2: The ω-, α-, β-c.c. system (1) is uniquely solvable in l if and only if: either (i) there exists m such that $n \geqq m$ implies $\Delta_n \neq 0$ and $\sup \|(E - B_n)^{-1}\| < \infty$ $(m \leqq n < \infty)$; or (ii) there exists m such that $\Delta_m \neq 0$ and $\|(E - B_m)^{-1}\| \cdot \|A - B_m\| < 1$. Theorem 3: If the ω-c.c. system (1) is uniquely solvable in l, then the sequence

$$\left\{ X_n = \left(\frac{\Delta_n^1(H)}{\Delta_n}, \frac{\Delta_n^2(H)}{\Delta_n}, \cdots, \frac{\Delta_n^n(H)}{\Delta_n}, h_{n+1}, h_{n+2}, \cdots \right) \right\}$$

$$(n = m, m+1, \cdots)$$

converges in l to the solution X. Here $\Delta_n^k(H)$ is obtained from Δ_n by replacing the kth column by $h_1, \cdots, h_n$.

I. M. Sheffer (University Park, Pa.)

1820:

Kreiss, Heinz-Otto. Über Matrizen die beschränkte Halbgruppen erzeugen. Math. Scand. **7** (1959), 71–80.

This paper is concerned with metric properties of complex square matrices of a fixed order. A^* denotes the transposed conjugate of A; I denotes the unit matrix; and the matrix norm $\|\cdot\|$ is defined by the relation

$$(*) \qquad \|A\| = \sup \frac{\|Au\|}{\|u\|} \quad (u \neq 0),$$

where $\|u\|$ is the euclidean norm of the vector u. The problem discussed is as follows: given a family $\mathfrak{F}$ of matrices, what are necessary and sufficient conditions for the matrices e^{At} to be uniformly bounded, for $A \in \mathfrak{F}$ and $t \geqq 0$, with respect to $\|\cdot\|$? When all matrices in $\mathfrak{F}$ are normal, then the requirement that all characteristic roots of all matrices in $\mathfrak{F}$ should have non-positive real parts is easily seen to be both necessary and sufficient. In the general case, this condition is still necessary but no longer sufficient. However, the author determines three sets of conditions each of which is equivalent to the original requirement of uniform boundedness. (For brevity, we shall only state two of them.) He proves, in fact, the equivalence of the following statements relating to an arbitrary family $\mathfrak{F}$ of matrices. (i) There exists a constant C_1 such that $\|e^{At}\| \leqq C_1$ for all $A \in \mathfrak{F}$ and all $t \geqq 0$. (ii) There exists a constant C_2 such that, for all $A \in \mathfrak{F}$ and all complex numbers s with $\operatorname{Re} s > 0$,

$$\|(A - Is)^{-1}\| \leqq C_2 / \operatorname{Re} s.$$

(iii) There exists a constant C_3 so that, corresponding to every matrix $A \in \mathfrak{F}$, there is a positive definite hermitian matrix H with $\max(\|H\|, \|H^{-1}\|) \leqq C_3$ and such that all characteristic roots of the hermitian matrix $HA + A^*H$ are non-positive.

The results of the paper are stated to have useful applications in the treatment of Cauchy's problem for partial differential equations.

{Reviewer's remark. The author does not define $\|\cdot\|$ explicitly and calls it (a little misleadingly) the euclidean norm. However, he uses definition (*), as can be inferred from the argument. His theorem retains, of course, its validity for any choice of norm since, for any two norms $\|\cdot\|_1$ and $\|\cdot\|_2$, the ratio $\|A\|_1 / \|A\|_2$ lies between two positive constants which are independent of A [A. Ostrowski, Math. Z. **63** (1955), 2–18; MR **17**, 228; Satz IV].}

L. Mirsky (Sheffield)

1821:

Mewborn, A. C. Generalizations of some theorems on positive matrices to completely continuous linear transformations on a normed linear space. Duke Math. J. **27** (1960), 273–281.

Let K be a closed positive cone in a real Banach space S, and let A be a completely continuous linear operator in S that is positive with respect to K in the sense that $AK \subset K$ and $Ax - \alpha x \in K$ for some non-zero x in K and some positive α. Let λ be the largest characteristic value of A corresponding to a characteristic vector in K. It is proved that λ is the largest number μ such that there exists a non-zero vector u in K for which $Au - \mu u \in K$. If also A maps all non-zero points of K into the interior of K, then λ is the smallest number ν for which there exists a non-zero vector u in K for which $\nu u - Au \in K$. These results generalize an inclusion theorem of L. Collatz [Math. Z. **48** (1942), 221–226; MR **5**, 30] on matrices with positive elements. It is also proved that if P is a continuous linear projection such that $PK \subset K$ and $(I-P)K \subset K$, then the maximal characteristic value of PAP does not exceed the maximal characteristic value of A. This generalizes a theorem of Frobenius on the principal minors of a non-negative matrix. *F. F. Bonsall* (Newcastle-upon-Tyne)

1822:

Chacon, R. V.; Ornstein, D. S. A general ergodic theorem. Illinois J. Math. **4** (1960), 153–160.

The authors prove the following theorem, conjectured by E. Hopf. Let T be a positive linear operator on L_1, of norm less than or equal to 1. Then if f and p are functions in L_1, and p is non-negative, the limit

$$\lim_{n\to\infty} \sum_{k=0}^{n} T^k f \Big/ \sum_{k=0}^{n} T^k p$$

exists and is finite almost everywhere on the set

$$\{s \mid \sum_{k=0}^{\infty} (T^k p)(s) > 0\}.$$

The proof is based upon an ingenious extension and simplification of a fundamental "combinatorial" lemma of Hopf; part of the content of this lemma may be expressed as follows: If $f \in L_1$ and $\sup_{n\geq 0} (T^n f)(s) > 0$ on a set B, then $\int_B f^+ \geq \int_B f^-$. *J. T. Schwartz* (New York)

1823:

Segal, I. E. A theorem on the measurability of group-invariant operators. Duke Math. J. **26** (1959), 549–552.

A short proof of the following theorem is given. Let $U(\cdot)$ be a continuous unitary representation of a Lie group G on a Hilbert space H; let K be a subgroup of G such that the ring of operators (von Neumann algebra) $R = [U(a): a \in K]' \cap [U(a): a \in G]''$ (that is, the set of operators in the von Neumann algebra determined by the representation U which commute with the restriction of U to K) is of finite type; and let F be the subalgebra of the enveloping algebra of (the Lie algebra of) G consisting of elements invariant under K. Then the mapping $A \to \overline{dU(A)}$ (the closed operator corresponding to A in the representation U) from F into the closed operators in H is a *-homomorphism of F into the algebra of measurable operators affiliated with R. (Measurable operators are discussed in the author's previous paper [Ann. of Math. (2) **57** (1953), 401–457; Correction, (2) **58** (1953), 595–596; MR **14**, 991; **15**, 204].) Essentially the theorem asserts that each operator in F goes into an operator with a sufficiently large domain—e.g., if A is symmetric then $\overline{dU(A)}$ is self-adjoint—and furthermore that the operators may be freely added and multiplied without destroying this property. The theorem applies to the case of G semi-simple and K the maximal essentially compact subgroup.

The author points out an error, discovered by L. Gross, in a previous paper of his [Duke Math. J. **18** (1951), 221–265; MR **13**, 534]. In this paper it was incorrectly asserted that a symmetric operator of degree ≤ 2 in the enveloping algebra always goes into a self-adjoint operator. (The applications of this lemma made in the paper are correct since only central operators were considered.) The review of this paper also stands in need of correction, for the reviewer asserts that the (insufficient) hypothesis of degree ≤ 2 is "évidemment superflue" and gives a proof for the case of arbitrary degree based on the assumption that symmetric operators which are linear combinations of operators having Friedrichs extensions are self-adjoint. *E. Nelson* (Princeton, N.J.)

1824:

Naimark, M. A. ★Normed rings. Translated from the first Russian edition by Leo F. Boron. P. Noordhoff N. V., Groningen, 1959. xvi+560 pp. Dfl. 45.00, $12.00; cloth: Dfl. 48.75, $13.00.

According to the brief foreword by the author, this differs from the original [GITTL, Moscow, 1956; MR **19**, 870] only in the correction of misprints and inaccuracies. [Cf. #1825 below.]

1825:

Smithies, F. Extensions of ideals in associative algebras. Proc. Cambridge Philos. Soc. **55** (1959), 277–281.

Let A be an algebra without a unit and let A' be the algebra obtained by adjoining a unit to A. A right ideal I is regular if there is an element u in A such that $ux - x$ is in I, for each x in A; u is called a left relative unit. A right ideal I' of A' is called an extension of I if I' is not in A and $I' \cap A = I$. Both Loomis [*An introduction to abstract harmonic analysis*, Van Nostrand, New York, 1953; MR **14**, 883; p. 59] and Naĭmark [*Normirovannye kol'ca*, Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1956; MR **19**, 870; p. 150] state that I' is unique for a regular right ideal I. The author shows by means of an example that this is false in general. He then goes on to prove the following. Theorem 2: A regular right ideal I has a unique extension if and only if the difference of any two left relative units is in I. Theorem 4: If I is a regular maximal right ideal, then I' is unique. Theorem 5: If a two-sided ideal is regular as both a left and right ideal, then there is a two-sided ideal I' in A' that is both the unique right and left extension of I. Theorem 6: If a two-sided ideal is regular as a right ideal and has a unique right extension, then it is regular as a left ideal. It is pointed out that the error in the books cited was also caught by Hewitt in the review of Naĭmark's book cited above. {It was caught earlier by R. Arens in his review of Loomis's book in Bull. Amer. Math. Soc. **60** (1954), 279–281.} Hewitt's proof of theorem 4 appears on p. 117 of his article in *Some aspects of analysis and probability*, pp. 105–168 [Wiley, New York, 1958; MR **21** #2159]. Theorems 2 and 4 are also proved in the appendix

of N. Bourbaki, *Algèbre, Chap. 8: Modules et anneaux semi-simples* [Hermann, Paris, 1958; MR **20** #4576]. Finally, it should be noted that the English translation of Naĭmark's book [#1824 above; see p. 161] also corrects this error.

A. Rosenberg (Evanston, Ill.)

1826:

Yood, Bertram. Faithful *-representations of normed algebras. Pacific J. Math. **10** (1960), 345–363.

This paper is mainly concerned with the problems of deciding whether an Arens *-algebra is a B^*-algebra for an equivalent norm, and with the problem of the existence of faithful *-representations of Banach algebras with involution. In what follows we shall denote by B a Banach algebra with an involution $x \to x^*$. For each $x \in B$, $\mathrm{sp}(x)$ and $r(x)$ are the spectrum and the spectral radius of x, respectively. The algebra B is an Arens *-algebra if there is $k>0$ such that (A) $\|x^*x\| \geq k\|x^*\|\,\|x\|$ for each $x \in B$. The paper is divided into six paragraphs. The first is a general introduction. The main results of the second paragraph are the following. (1) Suppose that for each hermitian element $u \in B$, $\mathrm{sp}(u)$ is real and $r(u) \geq c\|u\|$ for a certain $c>0$ (as is shown in the paper this is always the case if B is an Arens *-algebra). Then (1.1) B is an Arens *-algebra and (A) is satisfied for $k=c^5/(1+c)(1+2c^2)$; (1.2) B is a B^*-algebra, for an equivalent norm, if c is strictly superior to the real root of the equation $4t^3-2t^2+t-1=0$. The result (1.2) confirms (partially) a conjecture of I. Kaplansky. The third paragraph is concerned with faithful *-representations. Let R_0 be the set of all finite sums of elements of the form xx^* and R the set of all elements $x \in B$ for which there is $y \in R_0$ such that $ty+(1-t)x \in R_0$ for $0<t\leq 1$. The author shows that (2) if B has an identity then B has a faithful *-representation if and only if $x \to x^*$ is continuous and $\bar{R}_0 \cap (-\bar{R}_0)=\{0\}$; (3) B has a faithful *-representation if and only if $R \cap (-R)=\{0\}$, and $x=0$ whenever x is hermitian and $r(x)=0$. The next paragraph contains some results to be used in the paragraph 5. In paragraph 5 is discussed the problem of the existence of faithful *-representations in the case when B is semi-simple and has minimal one-sided ideals; the case when B is only a normed algebra is also treated. In the last paragraph the author characterizes the involutions (they have the form $T \to U^{-1}T^*U$ where U is hermitian and invertible) and the "proper" involutions on $L(H, H)$, where H is a Hilbert space.

C. Ionescu Tulcea (New Haven, Conn.)

1827:

Schaefer, H. H. On nonlinear positive operators. Pacific J. Math. **9** (1959), 847–860.

Let E be a locally convex topological linear space, C a closed proper cone of E, and T a continuous (not necessarily linear) operator defined on a neighbourhood U of 0 in C with the properties: (i) $T(U)$ is in C and relatively compact; (ii) $\overline{T(A)} \ni 0$ implies $\bar{A} \ni 0$ for every $A \subset U$. Applying Tychonov's fix-point theorem, the author proves the existence of a non-zero eigen vector of T in each of the following two cases. (1) There exists a continuous seminorm P such that $P(Tx) \neq 0$ whenever $Tx \neq 0$. (2) There exists $\sigma>0$ such that $T(\lambda x)=\lambda^\sigma Tx$ for every $\lambda>0$. (This corrects Satz 3.1 of the previous paper of the author [Math. Ann. **129** (1955), 323–329; MR **17**, 62].)

Applications to an algebraic integral equation and a nonlinear integro-differential equation are discussed.

I. G. Amemiya (Tokyo)

CALCULUS OF VARIATIONS

See also B1981.

1828:

Cinquini, Silvio. Un'osservazione sopra la figurativa degli integrali quasi-regolari seminormali. Boll. Un. Mat. Ital. (3) **14** (1959), 321–326. (French summary)

Résumé de l'auteur: "Quel que soit le nombre (fini) des dimensions de l'espace où est renfermé la (hypersurface) figurative, si l'intégral est 'quasi-regolare seminormale', sur cette figurative il n'y a aucune ligne droite."

W. H. Fleming (Providence, R.I.)

GEOMETRY

See also 1557.

1829:

Eisenhart, Luther Pfahler. ★Coordinate geometry. Dover Publications, Inc., New York, 1960. xi+298 pp. Paperbound: $1.65.

Unaltered republication of the first edition [Ginn and Co., Boston, 1939] of this analytic geometry textbook, including the appendix on equivalence with Hilbert's synthetic geometry.

1830:

Hartley, E. M. ★Cartesian geometry of the plane. Cambridge University Press, New York, 1960. xi+324 pp. $3.75.

From the introduction: "A first course in coordinate geometry ... for those working for the Advanced and Scholarship levels of the General Certificate of Education."

1831:

Neerup, P. O. F. Bachmann's axiomatic foundation of geometry. I. Nordisk Mat. Tidskr. **7** (1959), 97–110, 144. (Danish. English summary)

Verf. setzt sich das Ziel, die Grundgedanken des Aufbaus der Geometrie aus dem Spiegelungsbegriff (vgl. das unter diesem Titel erschienene Buch des Ref. [Springer, Berlin, 1959; MR **21** #6557]) und die darin liegende Algebraisierung der Elementargeometrie für einen grösseren Kreis darzustellen. Nach einer Einleitung, in der er sich kritisch mit der euklidischen Tradition auseinandersetzt, legt Verf. in dieser ersten Mitteilung zunächst Elemente der Gruppentheorie dar. Sodann diskutiert er, von einem intuitiven Standpunkt aus, Eigenschaften von Bewegungen in euklidischen, hyperbolischen und elliptischen Ebenen soweit, dass das gruppentheoretisch formulierte Axiomensystem der ebenen absoluten Geometrie [loc. cit. § 3] einsichtig wird. *F. Bachmann* (Kiel)

1832:

Dragomir, Achim. Remarques sur les coordonnées projectives. Lucrăr. Şti. Inst. Ped. Timişoara. Mat.-Fiz. **1958**, 145–148 (1959). (Romanian. French and Russian summaries)

On p. 27 of L. Bieberbach, *Projektive Geometrie* [Teubner, Berlin, 1931] there is a discussion of the Nagel point

of a triangle, the point of intersection of the lines joining the vertices of a triangle to the points of tangency of the exinscribed circles with the opposite sides. The author points out that Bieberbach made two errors in computation, in one using a condition for parallelism when he meant to use a condition for perpendicularity. A correct discussion is furnished. *A. Schwartz* (New York)

1833:

Tammi, Olli. On the use of the Schmid-Eckhart method in the theory of parallel projection. Ann. Acad. Sci. Fenn. Ser. A. I. No. 269 (1959), 17 pp.

Der Verfasser führt die Parameter der Schmid-Eckhartschen Figur (Gesamtschnittverfahren) ein, und ermittelt Zusammenhänge zwischen denselben. Mit Hilfe dieser Zusammenhänge gibt er dann eine allgemeine Behandlung der Aufgabe der Parallelprojektion des rechtwinkligen Koordinatensystems auf die Ebene. Endlich werden einige Spezialfälle der erwähnten Parallelprojektion mit Hilfe der Schmid-Eckhartschen Figur dargestellt.

L. Gyarmathi (Debrecen)

1834:

Miller, Maximilian. Der räumliche Rückwärtsschnitt und seine Ausgleichung. Wiss. Z. Hochsch. Verkehrswes. Dresden **3** (1955), 1–6. (Russian, English and French summaries)

Der Verfasser führt die räumliche Rückschnittaufgabe auf eine ebene Verschnittaufgabe zurück, wodurch die Docksche Lösung des erwähnten Problems [Österreich. Z. Vermessgswes. 8 (1910)] an praktischer Verwendbarkeit noch gewinnt. *L. Gyarmathi* (Debrecen)

1835:

Miller, Maximilian. Metrische Beziehungen in baryzentrischen ebenen Koordinaten und deren praktische Anwendung. Wiss. Z. Hochsch. Verkehrswes. Dresden **3** (1955), 105–110. (Russian, English and French summaries)

Nach einer kurzen Erörterung der historischen Entwicklung des baryzentrischen Kalküls drückt der Verfasser die Winkelfunktionen des Grunddreiecks mit Hilfe baryzentrischer Koordinaten aus. Die gewonnenen Formeln lassen sich vorteilhaft zur Lösung der ebenen Rückschnittaufgabe, sowie zur Bestimmung des Vialzentrums dreier gegebener Punkte verwenden. *L. Gyarmathi* (Debrecen)

1836:

Miller, Maximilian. Gesetzmässigkeiten bei der Teilung n-dimensionaler Räume und Kugelräume. Wiss. Z. Hochsch. Verkehrswes. Dresden **4** (1956), no. 1, 115–118. (English, Russian and French summaries)

Nach einigen historischen Bemerkungen über das Problem geht der Verfasser von dem durch Jakob Steiner gelösten Aufteilungsproblem des Euklidischen Raumes, der Ebene und der Sphäre aus [Crelles Journal **1** (1826), 349–364], und verallgemeinert die Steinerschen Resultate für den Fall des n-dimensionalen euklidischen Raumes. Bei der Lösung des Problems wird die Newtonsche "Differenzenbeziehung" verwendet. Auch auf die Zusammenhänge zwischen den einzelnen Aufgaben wird hingewiesen.

L. Gyarmathi (Debrecen)

1837:

Sedmak, Viktor. Sur quelques propriétés non topologiques ainsi que non métriques d'un complexe polyédrique. Rad Jugoslav. Akad. Znan. Umjet. **314**, Odjel Mat. Fiz. Tehn. Nauke **7**, 263–299 (1959). (French and Croatian)

The author considers various definitions for the dimension of a lattice [B. Dushnik, and E. W. Miller, Amer. J. Math. **63** (1941), 600–610; MR **3**, 73; H. Komm, ibid. **70** (1948), 507–520; MR **10**, 22; S. Ginsburg, ibid. **76** (1954), 590–598; MR **15**, 943]. He applies them to the uniform tessellations of the Euclidean plane [J. Kepler, *Gesammelte Werke*, Vol. 6, Beck, Munich, 1940, p. 73, figures D, E, F, L, N, P, S, V, Ii, Mm] and to the cubic honeycomb in Euclidean n-space.

H. S. M. Coxeter (Toronto)

1838:

Ladopoulos, Panaïotis. La géométrie euclidienne des éléments d'un réseau de coniques. C. R. Acad. Sci. Paris **250** (1960), 811–812.

The author shows how a projective geometry can be constructed whose "lines" are the conics of a net and whose "points" are the point-quadruples common to two conics of the net. A euclidean plane geometry is obtained by defining segments and their congruence.

L. M. Blumenthal (Columbia, Mo.)

1839:

Marchaud, André. Convexité et connexité linéaire. C. R. Acad. Sci. Paris **249** (1959), 2141–2143.

An extension to projective n-space of the author's previous work in projective 3-space [C. R. Acad. Sci. Paris **248** (1959), 2843–2844; MR **21** #3794; see review for terminology] is announced. The extension can be summarized in the following theorem. If two complementary sets in projective n-space ($n>1$) are each non-flat and linearly connected, then either (a) the closures of the two sets are degenerate convex domains and their common frontier is a system of two linear manifolds of $n-1$ dimensions, or (b) one only of the closures is a convex domain, degenerate or not, or (c) the common frontier is a ruled quadric of $n-1$ dimensions, degenerate or not.

L. M. Kelly (E. Lansing, Mich.)

1840:

Barlotti, A. Sulle possibili configurazioni del sistema delle coppie punto-retta (A, a) per cui un piano grafico risulta (A, a)-transitivo. Convegno internazionale: Reticoli e geometrie proiettive, Palermo, 25–29 ottobre 1957; Messina, 30 ottobre 1957, pp. 75–78. Edito dalla Unione Matematica Italiana con il contributo del Consiglio Nazionale delle Ricerche. Edizione Cremonese, Rome, 1958. vii+141 pp. 1800 Lire.

Verf. berichtet das Ergebnis seiner in Boll. Un. Mat. Ital. (3) **12** (1957), 212–226 [MR **19**, 674] erschienenen Arbeit, in der er eine Typeneinteilung der projektiven Ebenen nach der Menge der Paare (A, a) gegeben hatte, für welche die Gruppe der zentralen Kollineationen mit Zentrum A und Achse a transitiv ist. Er macht noch einige ergänzende Bemerkungen über die Frage nach der Existenz von Ebenen der verschiedenen möglichen Typen: Nach San Soucie [Proc. Amer. Math. Soc. **6** (1955), 291–296; MR **16**, 896] ist ein distributiver Quasikörper mit Rechtakürzungsregel für die Multiplikation bereits ein

Alternativkörper. Daher gibt es keine projektive Ebene vom Typ VI der Lenzschen Klasseneinteilung. Damit sei die Existenzfrage für alle Typen der Lenzschen Einteilung mit Ausnahme der Klasse III geklärt. {Bemerkung des Ref.: Das von Moulton stammende Beispiel einer projektiven Ebene (siehe Hilbert, *Grundlagen der Geometrie*, Teubner, Stuttgart, 1956 [MR 18, 227], § 23) ist vom Typ III, 2: Es gibt ein nicht inzidentes Paar (A, a) so daß die Ebene (A, a)-transitiv und (X, XA)-transitiv für jedes $X \in a$ ist. Von den 8 Untertypen der Klasse I können die Fälle 5, 7, 8 nicht vorkommen. Damit verbleiben insgesamt 19 Möglichkeiten; für 15 davon kennt Ref. Beispiele.}

H. Salzmann (Zbl 81, 370)

1841:

Panella, Gianfranco. Isomorfismo tra piani di traslazione di Marshall Hall. Ann. Mat. Pura Appl. (4) 47 (1959), 169–180.

Riassunto dell'autore: "Si studia la suddivisione in classi di equivalenza, rispetto alla relazione di isomorfismo geometrico, dell'insieme dei quasicorpi di M. Hall il nucleo dei quali non sia ismorfo a un suo sottocampo proprio. Si determina, inoltre, il gruppo delle collineazioni spettante ai piani di traslazione che rappresentano quelle classi."

1842:

Wagner, A. On perspectivities of finite projective planes. Math. Z. 71 (1959), 113–123.

The following result is due to Gleason [Amer. J. Math. 78 (1956), 797–807; MR 18, 593] and to André [Arch. Math. 6 (1954), 29–32; MR 16, 739]. Let π be a finite projective plane and G a collineation group of π; if for every point C of π and line c of π, with C on c [C not on c], there is a non-trivial (C, c) elation [homology], then π is Desarguesian and G contains the little projective group of π. The author shows here that the hypotheses can be weakened somewhat, with the same or similar results. Thus if it is assumed instead that every point of π is the center of a non-trivial elation, and every line of π is the axis of a non-trivial elation, it follows that π is Desarguesian. But if "elation" is replaced by "homology" in the previous sentence, it only follows that π is a translation plane or the dual of a translation plane. Examples show that π need not be Desarguesian. A corollary is that if a finite plane has a collineation group transitive on points, and if the plane has a non-trivial perspectivity (=elation or homology), then the plane is Desarguesian.

D. R. Hughes (Ann Arbor, Mich.)

1843:

Ostrom, T. G.; Wagner, A. On projective and affine planes with transitive collineation groups. Math. Z. 71 (1959), 186–199.

In zwei früheren Arbeiten [Canad. J. Math. 8 (1956), 563–567; 9 (1957), 389–399; MR 19, 445] hatte Ostrom gezeigt, daß eine endliche affine Ebene, deren Ordnung n nicht quadratisch und entweder ungerade oder eine (ungerade) Potenz von 2 ist, und die eine auf den affinen Punkten mindestens zweifach transitive Kollineationsgruppe besitzt, eine Translationsebene, d.h., eine Ebene über einem Quasikörper ist, und daß die Ebene sogar desarguessch ist, wenn ihre Kollineationsgruppe transitiv auf den Punktepaaren der zugehörigen projektiven Ebene ist. In der vorliegenden Arbeit werden diese beiden Ergebnisse ohne jede Voraussetzung über die Ordnung n der Ebene (außer der Endlichkeit) bewiesen. Dabei wird nur eine Fallunterscheidung gemacht, je nachdem ob n gerade oder ungerade ist, so daß die Beweise die vorher bekannten Spezialfälle mit umfassen. In Ergänzung und unter wesentlicher Benutzung eines früheren Satzes von A. Wagner [J. London Math. Soc. 33 (1958), 25–33; MR 20 #249], daß jede endliche viereckstransitive projektive Ebene desarguessch ist, zeigen die Verf. ferner, daß eine endliche affine Ebene, deren Kollineationsgruppe transitiv auf der Menge der nicht ausgearteten Vierecke mit zwei uneigentlichen Punkten operiert, desarguessch ist. Beim Beweis des Hauptresultats über die Gültigkeit des desarguesschen Satzes in jeder zweifach transitiven endlichen projektiven Ebene wird mehrfacher Gebrauch von einer vorhergehenden Arbeit von A. Wagner [#1842] gemacht, in der bewiesen wurde, daß eine endliche projektive Ebene, in der jeder Punkt und jede Gerade als Zentrum bzw. Achse einer Translation vorkommt, desarguessch ist, und daß sie eine Translationsebene oder das Duale davon ist, wenn jeder Punkt und jede Gerade Zentrum oder Achse einer Streckung ist. Mit Hilfe der dortigen Überlegungen folgt weiter, daß jede zweifach transitive Kollineationsgruppe einer endlichen projektiven Ebene die (von den Translationen erzeugte) kleine projektive Gruppe enthält und also sogar transitiv auf den einfach ausgearteten Vierecken ist; vgl. N. S. Mendelsohn [Canad. J. Math. 8 (1956), 532–562; MR 18, 665]. Als Zwischenergebnis hat man noch den Satz, daß für die Ordnung m der Fixunterebene einer 2-Gruppe von Kollineationen einer endlichen projektiven Ebene der Ordnung n eine Beziehung $n = m^{2^s}$ gilt. Schließlich werden endliche affine Ebenen mit einer auf den affinen Geraden transitiven Kollineationsgruppe betrachtet. Die Vermutung, daß solche Ebenen immer Translationsebenen sind, wird bestätigt, wenn die Ordnung n der Ebene gerade ist oder ungerade und kein Quadrat oder eine beliebige Primzahlpotenz.

H. Salzmann (Frankfurt a.M.)

1844:

Palman, Dominik. Vollkommen zirkuläre Kurven 3. Ordnung in der hyperbolischen Ebene. Glasnik Mat.-Fiz. Astr. Društvo Mat. Fiz. Hrvatske. Ser. II 14 (1959), 19–74. (Serbo-Croatian summary)

This is a rather elaborate and detailed study of the analogue in the hyperbolic plane of the complete circular curve of the third order in the complex euclidean plane. For the corresponding euclidean theory see the classic treatise of Loria [*Spezielle algebraische und transcendente ebene Kurven*, 2nd ed., Teubner, Leipzig, 1910–11]. Since the concepts of foci and circularity in the euclidean setting depend on the circular points, new definitions of these concepts must be formulated here. We list a few of these analogues to give some idea of the spirit of the study.

A curve of the third order is defined (after Von Staudt) as the intersection of corresponding members of a family of point conics (through four points) and a pencil of lines, the correspondence being a projectivity between the two forms. Isotropic lines in noneuclidean geometry are tangents to the absolute, foci of a curve are the intersection points of common tangents to the curve and the absolute. Thus, for example, a curve of the sixth class in noneuclidean geometry has 66 foci and in general a curve of the mth class has $m(2m-1)$ foci. A conic which cuts the

absolute in four pairwise different points is called a central conic, one which cuts it in two points and is tangent at another is a parabola and one tangent to the absolute in two distinct points is a circle. A circular curve is one tangent to the absolute, the degree of circularity being the number of such points of tangency (with higher order contacts being appropriately counted). Thus a curve of nth order has at most circularity degree n. If it has circularity degree n it is called complete.

The numerous theorems are too involved to quote here and are in large measure analogues of the classical theorems listed in Loria, loc. cit. We content ourselves with a listing of the section headings: 1. Circularity of curves. 2. Complete circular curves of third order. 3. Triaxial complete circular curves of third order. 4. Complete circular strophoids of third order. 5. Unicursal complete circular curves of third order.

L. M. Kelly (E. Lansing, Mich.)

1845:

Blaschke, Wilhelm. Sulle congruenze rettilinee nello spazio ellittico. Ann. Mat. Pura Appl. (4) **48** (1959), 209–221.

Some aspects of the geometry of the straight line in elliptic space E_3 with real or complex coordinates are developed in terms of quaternions. In the brief first section containing the basic definitions concerning quaternions it is stated that "i quaternioni furono introdotti nel 4.5.1748 da Eulero, usati da Gauss 1819 e ritrovati da Hamilton 1840". In § 2 a straight line in E_3 is defined by a pair of unit quaternions X, X', the distance formula and the formulae for elliptic motions are established. {Reviewer's remark: For a similar treatment see H. S. M. Coxeter; Amer. Math. Monthly **53** (1946), 136–146 [MR 7 387], § 10.} In § 3 the polar tetrahedron of the absolute $X\bar{X}=0$ is introduced. The remainder of the paper is concerned with the differential geometry of line congruences in E_3, normal congruences, Tshebyshev congruences, isotropic congruences.

H. Schwerdtfeger (Montreal)

CONVEX SETS AND GEOMETRIC INEQUALITIES

See also 1839.

1846:

Heppes, Aladár. On characterisation of curves of constant width. Mat. Lapok **10** (1959), 133–135. (Hungarian. Russian and English summaries)

Author's summary: "The author proves the following theorem: A closed convex plane curve is of constant width, if and only if every chord of it is a greatest chord of one of the two arcs determined by its endpoints."

1847:

Fejes Tóth, László. Über einem Kreis ein- und umbeschriebene Vielecke. Mat. Lapok **10** (1959), 23–25. (Hungarian. Russian and German summaries)

Zusammenfassung des Autors: "Es sei R ein regelmässiges n-Eck vom Inkreis k und Umkreis K, s ein k umbeschriebenes, S ein K einbeschriebenes n-Eck, s^* der Durchschnitt von s und K und S^* die konvexe Hülle von S und k. Es wird gezeigt, dass $T(s^*)\geqq T(R)$ und $L(S^*)\leqq L(R)$, wo T und L^* Flächeninhalt und Umfang bedeuten. Dagegen gelten die Ungleichungen $L(s^*)\geqq L(R)$ und $T(S^*)\leqq T(R)$ im allgemeinen nicht."

1848:

Mordell, L. J. On a distance-function inequality and A. Brauer's inequality for Cassini curves. J. Analyse Math. **6** (1958), 177–181.

A. Brauer's lemma on Cassini ovals [Duke Math. J. **19** (1952), 75–91; MR **13**, 813], which was used to find bounds for characteristic roots of matrices, states that for real a, b, c, k with $a<c<k$ and $b\leqq c$ the set of points in the z plane with $|z-b|\,|z-c|\leqq(k-b)(k-c)$ lies in the open domain $|z-b|\,|z-a|<(k-b)(k-a)$ with the exception of the point $z=k$ which is the only common point of the boundaries of both sets. This theorem is here generalized in several ways. One generalization states that if $\lambda_1, \cdots, \lambda_n, \lambda_{n+1}, \lambda'_{n+1}$ are real numbers all $<k$ and none exceeding λ_{n+1}, and $\lambda'_{n+1}\leqq\lambda_{n+1}$, then

$$\prod_{r=1}^{n+1}|z-\lambda_r| \leqq \prod_{r=1}^{n+1}(k-\lambda_r)$$

implies

$$\prod_{r=1}^{n}|z-\lambda_r|\,|z-\lambda'_{n+1}| \leqq \prod_{r=1}^{n}(k-\lambda_r)(k-\lambda_{r+1}).$$

Equality arises only when $z=k$. A wider generalization deals with similar inequalities in a finite-dimensional metric space.

L. Moser (Edmonton, Alta.)

1849:

Groemer, Helmut. Über die Einlagerung von Kreisen in einen konvexen Bereich. Math. Z. **73** (1960), 285–294.

The following result is proved. Let B be a convex region in the plane, of area F and circumference U. If n circles of unit radius can be placed in B without overlapping, then $n\sqrt{12}\leqq F-\kappa U+\lambda$, where $\kappa=(2-\sqrt{3})/2=0.1339\cdots$ and $\lambda=\sqrt{12}-\pi(\sqrt{3}-1)=1.1642\cdots$. Equality holds exactly when B is the convex hull of the enclosed circles and when, in addition, the convex hull H of the centres of these circles either can be subdivided into equilateral triangles of side 2 where the triangle vertices are centres of circles, or H can be subdivided into line segments of length 2 both endpoints of which are centres of circles; or H is a single point. This theorem strengthens the inequality $n\sqrt{12}\leqq F$ where $n\geqq 2$, which is due to Fejes Tóth [*Lagerungen in der Ebene, auf der Kugel und im Raum*, Springer, Berlin, 1953; MR **15**, 248; p. 67]. The proof is rather involved.

K. Mahler (Manchester)

1850:

Melzak, Z. A. A class of star-shaped bodies. Canad. Math. Bull. **2** (1959), 175–180.

From the author's introduction: "The more important properties of the class $\mathscr{K}$ of all bounded convex bodies in E_3 with nonempty interior include uniform approximability by polyhedra, existence of volume and surface area, and Blaschke's selection principle. In this note we define and consider a class $\mathscr{H}$ of star-shaped bodies in E_3 which enjoys many properties of $\mathscr{K}$, among them the above-mentioned ones, and is considerably larger. Roughly speaking, $\mathscr{H}$ consists of closed bounded sets in E_3 with

nonempty interior, whose boundary is completely visible from every point of a set with nonempty interior. It turns out that $\mathscr{H}$ is identifiable with the class of all real-valued positive functions on the sphere S_3 which satisfy a Lipschitz condition."

Now let $K \in \mathscr{H}$ and denote by $B(K)$ the boundary of K, by $H(K)$ the set of all points from which K is starshaped. Let $P(K)$ denote the set of all points $p \in B(K)$ such that $B(K)$ admits a tangent plane π_p at p. Let $o \in H(K)$ and (for $p \in P(K)$) denote by $U(\pi_p)$ the closed halfspace which contains o and is bounded by π_p. The author proves that $H(K) = \bigcap_{p \in P(K)} U(\pi_p)$. A set $Y \subset B(K)$ is called "enveloping" provided $H(K) = \bigcap_{p \in Y} U(\pi_p)$, and "essential" if it is enveloping, closed, and possesses no proper enveloping subset. The author proves that if $K \in \mathscr{H}$, $H(K)$ is interior to K, $B(K)$ is of class C^2, and the sets of elliptic and hyperbolic points of $B(K)$ are both open in $B(K)$, then every essential subset of $B(K)$ consists entirely of parabolic points. *Victor Klee* (Seattle, Wash.)

DIFFERENTIAL GEOMETRY

See also 1845, 1926, 2202.

1851:

Sakellariou, Nilos. Spherocentrical lines of different classes of a given curve. Bull. Soc. Math. Grèce **29**, 25–33 (1954). (Greek. English summary)

Formulas concerning the Frenet vectors of $\gamma^{(i)}$, where $\gamma = \gamma^{(0)}$ is a curve in E^3 and $\gamma^{(i+1)}$ is the locus of the center of the osculating sphere of $\gamma^{(i)}$.

1852:

Gołąb, S.; Kordylewski, J.; Kuczma, M. On some new geometrical interpretations of the torsion of a skew curve. Ann. Polon. Math. **7** (1960), 269–278.

Two geometric interpretations of the torsion of a curve in Euclidean 3-space are given. The proofs are straightforward but involve lengthy calculations. The simpler of these results is the following: Let C and C' be a skew curve and a plane curve respectively which at all corresponding points P and P' have equal arc length parameters and equal first curvatures κ. If Q and Q' are nearby corresponding points of the curves where arc PQ = arc $P'Q' = s$ and the difference chord PQ − chord $P'Q' = d$, then the magnitude of the torsion τ of C is

$$|\tau| = \lim_{s \to 0} \kappa^{-1} \sqrt{(720 d/s^5)}.$$

A. Fialkow (Brooklyn, N.Y.)

1853:

Drăgilă, Pavel. Sur les couples de surfaces harmoniques conjuguées. Acad. Roy. Belg. Bull. Cl. Sci. (5) **45** (1959), 726–733.

Cet article prolonge des études antérieures, dues à différents auteurs, sur les surfaces harmoniques et plus spécialement les surfaces harmoniques conjuguées, et leur apporte d'intéressants compléments. Il montre que les relations de parallélisme $\vec{M}'_{u'} = k\vec{M}_v$, $\vec{M}'_{v'} = h\vec{M}_u$ entre deux surfaces (M) et (M') rapportées à des coordonnées curvilignes (u, v), (u', v') conduisent aux couples de surfaces harmoniques conjuguées, présentant ainsi sous forme géométrique les fonctions harmoniques et les fonctions analytiques. Étudiant la possibilité de trouver sur les deux surfaces d'un même couple supposées rapportées aux mêmes paramètres u, v, des systèmes de nouveaux paramètres curvilignes $[u = \varphi(s, t)$, $v = \psi(s, t)]$ vérifiant les relations (dites de parallélisme ponctuel): $\vec{M}'_s = k\vec{M}_t$, $\vec{M}'_t = h\vec{M}_s$, l'auteur montre qu'il existe une infinité de tels systèmes, déterminés par l'équation $u_s u_t + v_s v_t = 0$, laquelle admet des solutions qui sont harmoniques et d'autres qui ne le sont pas. Cherchant à disposer des paramètres s et t de manière que les tangentes homologues soient parallèles sur l'un des couples envisagés, il obtient deux réseaux en correspondance de Koenigs sur les deux surfaces. Il envisage ensuite les surfaces harmoniques sur lesquelles les courbes u, v forment un réseau conjugué, et montre que les foyers des congruences formées par les droites joignant les points homologues de deux surfaces vérifiant $\vec{M}'_u = -\vec{M}_v$, $\vec{M}'_v = \vec{M}_u$ partagent harmoniquement les segments (MM'). *P. Vincensini* (Marseille)

1854:

Arghiriade, E. Réseaux et congruences d'ordre supérieur. Acad. R. P. Romîne. Stud. Cerc. Mat. **9** (1958), 7–111. (Romanian. Russian and French summaries)

This paper, covering more than a hundred pages, aims at giving a systematical study of nets and congruences of higher order on a surface of a projective space. The author outlines the course of successive generalizations which led from the notion of ordinary conjugate nets to his generalized net of order (p, q). This latter arises from the study of the equation

$$(*) \qquad \sum_{i=0}^{p} \sum_{j=0}^{q} a_{ij} x^{ij} = 0 \quad (x^{ij} = \partial^{i+j} x / \partial u^i \partial v^j),$$

called the generalized Laplace equation of order (p, q). The curves $du = 0$, $dv = 0$ on the surface $x(u, v)$ are said to form a generalized net of order (p, q) if $x(u, v)$ satisfies (*), which reduces, taking $p = 1$, $q = 1$, to the case of ordinary nets. The matrix Σ_0 with elements x^{ij} defines a linear space, called the fundamental Laplace space. From this the author deduces, applying a version of Segre's method, two sequences of spaces $\{\Sigma_\alpha^u\}$ $(\alpha = 0, \cdots, p+1)$, $\{\Sigma_\beta^u\}$ $(\beta = p+2, \cdots, r)$, which are called principal resp. secondary Laplace spaces, taken in the direction u. The properties of these spaces are discussed in detail.

The case $(p, 1)$ turns out to be especially important due to the property that the dimensions of the spaces figuring in the sequence decrease by unity. By analogy a net of order (p, q) is an S_u net if the dimensions of the secondary spaces decrease by unity. The main properties of a subset of the S_u nets (non-particular nets) are announced and proved.

The next notion generalized by the author is that of congruence. By definition the line xx_1 is said to describe a generalized congruence of order (p, q) with x and x_1 as foci if $x(u, v)$ and $x_1(u, v)$ satisfy

$$x^{0q} = \sum_{i=0}^{q-1} a_i x^{0i} + \sum_{i=0}^{p-1} b_i x^{i0}, \quad x_1^{p0} = \sum_{i=0}^{p-1} \alpha_i x^{i0} + \sum_{i=0}^{q-1} \beta_i x^{0i}.$$

The congruence is non-singular if some determinants deduced from the coefficients of the above equations are different from zero. Non-singular congruences have a series of interesting properties. The paper ends with an illustrative example. *G. Soós* (Debrecen)

1855:

Yuza, Miloslav [Jůza, Miloslav]. Quelques homographies associées à une correspondance entre deux courbes. Czechoslovak Math. J. 8 (83) (1958), 563–572. (Russian. French summary)

Soit $f: {}^1C(t) \to {}^2C(t)$ une correspondance entre deux courbes d'espace projectif S_n; ${}^iC(t)$ soit donnée par l'équation différentielle ${}^iC^{(n+1)} = \sum_{i=0}^{n} {}^ip_{n-i+1}\,{}^iC^{(i)}$. Pour chaque t on a une homographie $L(t)$ [ou $\Lambda(t)$, ou $K(t)$]: $S_n \to S_n$ qui réalise un contact analytique d'ordre $n+1$ entre ${}^1C$ et ${}^2C$ [ou est définie corrélativement, ou est définie par la propriété de posséder une enveloppe]; on a $K(t)\,{}^1C^{(i)} = {}^2C^{(i)}$ $(0 \leqq i \leqq n)$; si l'on suppose ${}^1p_1 = {}^2p_1$, la valeur $K_0(t)$ de $K(t)$ devient complètement déterminée par f. On étudie les relations qui existent entre $L(t)$, $\Lambda(t)$ et $K(t)$. *A. Švec* (Prague)

1856:

Stoka, M.; Vrănceanu, G. G. Correspondances entre deux espaces projectifs à caractéristiques confondues. Acad. R. P. Romîne. Stud. Cerc. Mat. 10 (1959), 219–235. (Romanian. Russian and French summaries)

Soit donnée une correspondance $u^i = u^i(x^j)$ entre deux espaces projectifs $A_3(x^i)$, $E_3(u^i)$, alors il existe un tenseur π^i_{jk} sur A_3 qui donne la connexion projective de A_3 pour lequel u^i sont les coordonnées cartésiennes; v. Vrănceanu, Boll. Un. Mat. Ital. (3) 12 (1957), 489–506 [MR 19, 1075]. Les droites caractéristiques de la correspondance sont données par les équations du troisième ordre (*) $f_3(y', z') = g_3(y', z') = 0$ où $y' = dx^2/dx^1$, $z^1 = dx^3/dx^1$. On trouve la forme du tenseur π^i_{jk} au cas où les courbes (*) possèdent des courbes communes et aucun point isolé commun ou elles se réduisent à une droite confondue; alors on considère les mêmes cas pour une connexion projective constante et l'on présente effectivement les correspondances en question. *A. Švec* (Prague)

1857:

Eisenhart, Luther P. The Ricci principal direction vectors. Proc. Nat. Acad. Sci. U.S.A. 45 (1959), 1028–1031.

The proofs of the following theorems are given: In a general V_n the Ricci principal direction vectors $\lambda_{\lambda}{}^{i}$ are an orthogonal ennuple. In a V_n for which the equation $R_{ij,l} = \frac{1}{2}R_{,i}g_{jl} + \frac{1}{2}R_{,j}g_{il}$ holds the vectors $\lambda_{\lambda}{}^{i}$ are normal, that is, each set of vectors $\lambda_{\lambda}{}^{i}$ is normal to a family of hypersurfaces. *A. Švec* (Prague)

1858:

Murgescu, Viorel. Propriétés conformes des espaces de Riemann généralisés. I. Bul. Inst. Politehn. Iași (N.S.) 4 (8) (1958), no. 3–4, 77–80. (Russian and Romanian summaries)

The author separates the tensor g_{ik} of a generalised riemannian space G_n into its symmetrical and alternating parts, $g_{ik} = g_{\underline{ik}} + g_{\underline{\underline{ik}}}$, and, considering two spaces G_n and $\dot{G}_n$ connected by the conformal relation $\dot{g}_{ik} = 2\sigma g_{ik}$, he eliminates σ by means of the analogue to the Christoffel symbols of the first kind $\Delta_{ikl} = \frac{1}{2}(\partial g_{il}/\partial x_k + \partial g_{kl}/\partial x_i - \partial g_{ik}/\partial x_l)$, obtaining three-index quantities having the same value at corresponding points—the coefficients of the conformal relation—

$$C_{ij}{}^{h} = \Delta_{ij}{}^{h} - \frac{1}{n}\left(\Delta_{\underline{i\alpha}}{}^{\alpha} g_{\underline{\underline{j}}}{}^{\underline{\underline{h}}} + \Delta_{\underline{j\alpha}}{}^{\alpha} g_{\underline{\underline{i}}}{}^{\underline{\underline{h}}} - \Delta_{\underline{\underline{i}}}{}^{\alpha} g_{\underline{j}}{}^{\underline{\underline{h}}}\right),$$

which leads to $C_{ik}{}^{h} = 0$, and finally gives the transformation equations of $C_{ij}{}^{h}$ explicitly.

E. M. Bruins (Amsterdam)

1859:

Nagano, Tadashi. On conformal transformations of Riemannian spaces. J. Math. Soc. Japan 10 (1958), 79–93.

A thorough investigation of conformal transformations is given in this paper, using many different kinds of tools; not just the clasical ones of tensor analysis, but also those of the theory of Lie groups, covering spaces, and several others. 'Space' means connected Riemann space of dimension $n > 3$. Very useful is the result (arising from a theorem of Hlavatý [Nederl. Akad. Wetensch. Proc. Ser. A 38 (1935), 281–286]) that if M is not conformally flat, then the conformal group $C(M)$ is isomorphic to the isometry group $I(M^*)$, where M^* is the open set of M of those points where the conformal curvature does not vanish, and the metric on M^* is a modification of that of M by a factor equal to the norm of the conformal curvature tensor. If M is conformally flat, the Lie algebra $C^I(M)$ of infinitesimal conformal transformations is isomorphic to a subalgebra of the n-Möbius algebra; and the n-sphere is the only space for which the two algebras coincide. Paralleling some investigations of Kuiper [Ann. of Math. (2) 50 (1949), 916–924; MR 11, 133] the universal covering space of a conformally flat space with transitive connected conformal transformation group is conformally the same as the universal covering space of an open submanifold of the sphere. Sections on transitive and intransitive conformal groups of higher dimensions, using the structure of n-Möbius algebras, give explicit classifications of the conformal groups in some cases.

A. Nijenhuis (Seattle, Wash.)

1860:

Yano, Kentaro. Some integral formulas and their applications. Michigan Math. J. 5 (1958), 63–73.

The author extends his arsenal of integral formulas with a few more; one adapted to conformal Killing vector fields, and one to (pseudo-)analytic vector fields in (pseudo-) Kählerian spaces. A total of 27 theorems are proved, of which some are included for completeness; others have different proofs, and some 17 do not contain references, so are presumably new. Several seem of considerable interest, such as relations established between eigenvalues of the Ricci tensor and bounds on λ in the equation $\Delta f = \lambda f$ in Kähler manifolds.

A. Nijenhuis (Seattle, Wash.)

1861:

Yano, Kentaro; Nagano, Tadashi. Some theorems on projective and conformal transformations. Nederl. Akad. Wetensch. Proc. Ser. A. 60 = Indag. Math. 19 (1957), 451–458.

Four sets of theorems are proved (total count 21), dealing with (1) projective transformations leaving invariant the covariant derivative of Weyl's projective curvature tensor; (2) conformal transformations leaving

invariant Weyl's conformal curvature tensor; (3) projective transformation in Einstein spaces; (4) conformal transformations in Einstein spaces.

A. Nijenhuis (Seattle, Wash.)

1862:

Couty, Raymond. Transformations infinitésimales projectives. C. R. Acad. Sci. Paris **247** (1958), 804–806.

A transformation of a Riemannian manifold is said to be projective if it maps every geodesic into a geodesic. The author studies an infinitesimal projective transformation X of a Riemannian manifold and derives the following new formula. Let ξ be the 1-form associated to X. (By means of the metric, every infinitesimal transformation can be identified with a 1-form.) Then

$$\Delta\xi-(2/n+1)d\delta\xi = Q\xi,$$

where $(Q\xi)_i=2\sum R_{ik}\xi^k$. Using this formula, the author proves: Theorem 1: On a compact (orientable) Riemannian manifold, there exists no infinitesimal projective transformation X satisfying $\sum R_{ij}\xi^i\xi^j\leq 0$ unless it is a parallel vector field and then automatically $\sum R_{ij}\xi^i\xi^j=0$. In particular, if the Ricci tensor is negative definite everywhere, the manifold admits no infinitesimal projective transformation. Theorem 2: Let V be an Einstein space with $R\neq 0$ (where R denotes the scalar curvature). Assume either that V admits no parallel vector field or that V is compact. Let $L=$ the Lie algebra of infinitesimal projective transformations, $L_1=$ the Lie algebra of Killing vector fields, and $L_2=\{X\in L;\ \xi \text{ is an exact 1-form}\}$. Then $L=L_1+L_2$ (direct sum as vector space), $[L_1, L_2]\subset L_2$, $[L_2, L_2]\subset L_1$. As an application of theorem 2, he obtains theorem 3: If V is a complete Einstein space with $R<0$, then every infinitesimal projective transformation of bounded length is a Killing vector field.

By a similar technique he obtains the following three theorems on Kaehler manifolds. Theorem 4: On a Kaehler manifold V_{2n} $(n>1)$, every infinitesimal projective transformation X whose associated 1-form ξ is closed is affine. Combining theorem 4 with a theorem of Yano [Ann. of Math. (2) **55** (1952), 38–45; MR **13**, 689] he obtains theorem 5: In theorem 4, assume furthermore that V_{2n} is compact. Then X is a parallel vector field. Finally he proves theorem 6: Every infinitesimal projective transformation on an Einstein-Kaehler manifold V_{2n} $(n>1)$ with $R\neq 0$ is a Killing vector field.

{Remarks by the reviewer: (1) Theorem 1 strengthens a result of Bochner [Bull. Amer. Math. Soc. **52** (1946), 776–797; MR 8, 230] and that of Yano and Nagano [#1861 above; in particular, th. 3.2]. (2) The assumption on holonomy and on compactness in theorem 2 seems unnecessary, because an Einstein space with $R\neq 0$ admits no parallel vector field. (3) Theorem 6 improves th. 3.3 and th. 3.4 of the paper of Yano and Nagano.}

S. Kobayashi (Cambridge, Mass.)

1863:

Kostant, Bertram. A characterization of invariant affine connections. Nagoya Math. J. **16** (1960), 35–50.

Ambrose and Singer [Duke Math. J. **25** (1958), 647–669; MR **21** #1628] proved that a complete simply connected Riemannian manifold is homogeneous if and only if there exists a tensor field S of type $\binom{i}{j\ k}$ satisfying certain conditions. The author points out here that the conditions on S imposed by Ambrose-Singer are equivalent to the covariant constancy of S and the Riemannian curvature tensor with respect to the affine connection $B=A+S$, where A denotes the Riemannian connection, and also to the covariant constancy of the torsion and curvature tensors of the affine connection B, and that, with this reformulation, the theorem of Ambrose-Singer may be obtained as a consequence of results of Nomizu and the reviewer on affine connections with covariant constant torsion and curvature. Then he generalizes the new formulation to the case of affine connection; he gives a necessary and sufficient condition for a simply connected manifold with affine connection A to be a reductive homogeneous space with respect to a connected Lie group leaving A invariant.

S. Kobayashi (Vancouver, B.C.)

1864:

Tsukamoto, Yôtarô. On certain Riemannian manifolds with positive sectional curvature. Mem. Fac. Sci. Kyushu Univ. Ser. A **13** (1959), 140–145.

Let M be a complete simply connected Riemannian manifold. The author proves that if the sectional curvature K of M satisfies the inequalities $0<hL\leq K\leq L$, where $h\sim 0.49$ is the solution of the equation $\sin\pi\sqrt{h}=(8/7)\sqrt{h}$, and L is a constant, then $d(p, c(p))\geq\pi/\sqrt{L}$ for all $p\in M$, where $c(p)$ is the cutlocus of p and $d(p, c(p))$ the distance between p and $c(p)$. This result is applied to show that M is homeomorphic with the sphere if $h\sim 0.54$ and M is a homological sphere if $h\sim 0.49$, a version somewhat different from those obtained by Rauch [Ann. of Math. (2) **54** (1951), 38–55; MR **13**, 159], Klingenberg [Math. Ann. **137** (1959), 351–361; MR **21** #4446] and Berger [C. R. Acad. Sci. Paris **247** (1958), 1165–1168; MR **20** #2754].

T. K. Pan (Norman, Okla.)

1865:

Berger, Marcel. Les variétés riemanniennes (1/4)-pincées. C. R. Acad. Sci. Paris **250** (1960), 442–444.

In this note the author announces the following improvements in the bounds of the theorem of Rauch for even-dimensional manifolds. This result may be stated precisely as follows. Let V be a compact simply connected even dimensional riemann manifold whose sectional curvature $\rho(u)$ has the property that $\delta\Delta\leq\rho(u)\leq\Delta$ $(\Delta>0)$ for all two-dimensional plane elements at all points of V. If $\delta>\frac{1}{4}$, then V is homeomorphic to the sphere of the same dimension. If $\delta=\frac{1}{4}$ and V is not homeomorphic to a sphere of the same dimension, then V is a compact symmetric riemann space of rank 1, with the usual metric.

L. Auslander (New Haven, Conn.)

1866:

Gohman, A. V. The intrinsic geometry of a rheonomic non-holonomic space. Dokl. Akad. Nauk SSSR **126** (1959), 255–258. (Russian)

The author studies a space in which the equations of motion of a point correspond to the equations of motion of a rheonomic dynamic system [cf. A. Wundheiler, Prace Mat.-Fiz. **40** (1932), 97–142]. A special rheonomic non-holonomic space is defined, some intrinsic geometric properties of such a space are explained and the necessary and sufficient conditions for its holonomity are established. Considering it as a Riemannian space with a common Riemannian connection the author formulates the D'Alembert principle and deduces from it the so-called Wundheiler equations.

T. P. Anđelić (Belgrade)

GENERAL TOPOLOGY, POINT SET THEORY

1867:

Burgess, C. E. Chainable continua and indecomposability. Pacific J. Math. **9** (1959), 653–659.

Some properties are derived for continua which are both linearly chainable (for every $\varepsilon>0$ there is a linear ε-chain (of open sets of diameter $<\varepsilon$) covering the space) and circularly chainable (the "first" and "last" element of the ε-chain intersect). E.g., a linearly chainable continuum M is circularly chainable if and only if it is either indecomposable or 2-indecomposable (that is, decomposable into two proper subcontinua but not non-trivially into three).

J. de Groot (Amsterdam)

1868:

Case, J. H.; Chamberlin, R. E. Characterizations of tree-like continua. Pacific J. Math. **10** (1960), 73–84.

The authors show that a one-dimensional continuum having no essential mapping into any linear graph is tree-like, i.e., every open covering has a refinement whose nerve is a tree. However, they exhibit an inverse limit of figure eights with essential coordinate projections but no essential mapping upon a circle; the example is also homologically trivial.

They attribute to the reviewer the conjecture that a homologically trivial one-dimensional continuum must be treelike. Actually the conjecture was made by P. S. Aleksandrov and the reviewer announced a proof.

J. Isbell (Seattle, Wash.)

1869:

Bing, R. H. A simple closed curve is the only homogeneous bounded plane continuum that contains an arc. Canad. J. Math. **12** (1960), 209–230.

The author proves the theorem stated in the title. In doing so, he establishes several properties for homogeneous compact metric continua which contain arcs. This leads to the rather natural question of classifying one-dimensional continua of this sort. None of them can be tree-like. The author sketches a proof that the circle-like ones are solenoids. However, this fact is not used in proving the main result from which it follows (without circular reasoning) that the dyadic (or higher) solenoid is not embeddable in the plane. *F. B. Jones* (Chapel Hill, N.C.)

1870:

Doyle, P. H. Tame triods in 3-space. Proc. Amer. Math. Soc. **10** (1959), 656–658.

The author shows that a triod T, i.e., the union of three topological arcs A_1, A_2, A_3 such that p is an endpoint of A_1, A_2, A_3 and $A_i \cap A_j=\{p\}$, $i\neq j$, is tamely imbedded in Euclidean 3-space if and only if every arc in T is tame and two arcs A_i, A_j lie in the interior of a disc (topological 2-cell) which intersects the remaining arc A_k only in the point p. The proof is based on theorem 2 of E. E. Moise [Ann. of Math. (2) **56** (1952), 96–114; MR **14**, 72]. By this theorem, a disc D with boundary S can be replaced by a disc D which is locally polyhedral mod S and is contained in an arbitrarily small neighborhood of D. {The remark in the first paragraph that the conditions in theorem 1 are independent is invalid, because of the difficulties in the author's previous paper, Duke Math. J. **26** (1959), 263–267; cf. the review MR **21** #5174.}

O. Masaitis (Aberdeen, Md.)

1871:

Stallings, John R. Uncountably many wild disks. Ann. of Math. (2) **71** (1960), 185–186.

By putting a horn on a solid ball in E^3, looping this horn, splitting it into two branch horns which are then linked with the loop, and continuing this process of looping, branching, and linking a countable infinity of times (after the first time the linking also produces the next loop), the author constructs a 3-cell Q whose bounding 2-sphere S contains a Cantor set C of horn tips at which S is not polyhedral. In fact any arc in Q running through infinitely many branches of a horn to its tip in C will be one of the classic type shown by Fox and Artin to be wild. The 3-cell Q contains a collection G of mutually exclusive disks each of which separates Q and contains one or two of the points of C. Obviously G has the cardinality of the continuum and the author shows that each element of G is wild.

This example is of particular interest since Bing has shown that no collection of mutually exclusive wild surfaces in E^3 is uncountable. So while E^3 cannot contain an uncountable collection of mutually exclusive wild 2-spheres, it does contain such a collection of wild disks.

F. B. Jones (Chapel Hill, N.C.)

1872:

Ryškov, S. S. On k-regular imbeddings. Dokl. Akad. Nauk SSSR **127** (1959), 272–273. (Russian)

A subset A of Euclidean n-space E^n is k-regular if no $k+2$ points of A are contained in a k-plane. A topological map f: $F\to E^n$ (F a compactum) is a k-regular imbedding if $f(F)$ is a k-regular subset of E^n. Borsuk posed the problem of finding the least n for which an arbitrary compactum of dimension m can be k-regularly imbedded in E^n, and conjectured that $n=2m+k$. Boltyanskiĭ proved that $n\leq mk+m+k$. The author shows that Borsuk's conjecture is false by exhibiting a class of compacta for which $n\geq[\frac{1}{2}(k+1)]m+[\frac{1}{2}k]$. *H. Komm* (Troy, N.Y.)

1873:

Papić, Pavle. Sur les espaces H-fermés. Glasnik Mat.-Fiz. Astr. Društvo Mat. Fiz. Hrvatske. Ser. II **14** (1959), 135–141. (Serbo-Croatian summary)

There is presented a Hausdorff space which is not H-closed, but in which each infinite set has a complete limit point. *R. Arens* (Los Angeles, Calif.)

1874:

Borsuk, Karol. Grundlagen der Geometrie vom Standpunkte der allgemeinen Topologie aus. The axiomatic method. With special reference to geometry and physics. Proceedings of an International Symposium held at the Univ. of Calif., Berkeley, Dec. 26, 1957–Jan. 4, 1958 (edited by L. Henkin, P. Suppes and A. Tarski), pp. 174–187. Studies in Logic and the Foundations of Mathematics. North-Holland Publishing Co., Amsterdam, 1959. xi+488 pp. $12.00.

Topological characterization of the classical spaces based upon a classification of topological spaces in general is a difficult and unsolved problem. The paper gives extensive references and a brief but clear survey of the results and some future possibilities in this field and related ones.

W. A. Wilson [Amer. J. Math. **54** (1932), 505–517] determined a set of metric axioms which characterize Euclidean n-space E^n metrically. The main axiom in this system is Menger's "four-point axiom" (every four points of the (metric) space are congruent to four suitable points in E^3). The author characterizes E^n by means of this metric "four-point axiom" and a set of purely topological axioms, actually by the axioms of a topological space, together with normality (why not regularity?), the existence of a countable base, local compactness, local connectedness, connectedness, n-dimensionality and—most important of all—two linking properties. *J. de Groot* (Amsterdam)

1875:

Borsuk, K. Theorie der Retrakte und ihre Probleme. Bŭlgar. Akad. Nauk Izv. Mat. Inst. **4**, no. 1, 45–54 (1959). (Russian and Bulgarian summaries)

A talk delivered to a meeting of Bulgarian mathematicians. The theory is outlined and the philosophy behind it is discussed. The six unsolved problems posed are concerned with dimension, Cartesian product and homotopy type of retracts and with imbedding.

R. H. Fox (Princeton, N.J.)

1876a:

Curtis, Philip C., Jr. A note concerning certain product spaces. Arch. Math. **11** (1960), 50–52.

1876b:

Gillman, Leonard. A P-space and an extremally disconnected space whose product is not an F-space. Arch. Math. **11** (1960), 53–55.

Curtis' paper reports results of W. Rudin, M. Henriksen, and the author. The product of two infinite compact spaces cannot be an F-space. [For definitions see Gillman and Henriksen, Trans. Amer. Math. Soc. **82** (1956), 366–391; MR **18**, 9.] Indeed, for a product to be F, both factor spaces must be F and one must be P. Curtis sharpens this result and obtains converses under various side conditions. However, the general converse is false, as Gillman shows with two door spaces which are respectively P and extremally disconnected (hence F).

J. Isbell (Seattle, Wash.)

1877:

Hayashi, Yoshiaki. On countably paracompact spaces. Bull. Univ. Osaka Prefecture Ser. A **7** (1959), 181–183.

Another proof of a result of Ishikawa [Proc. Japan Acad. **31** (1955), 686–687; MR **17**, 650]. No credit is given, even though Ishikawa's paper appears in the bibliography.

E. Michael (Princeton, N.J.)

1878:

Younglove, J. N. Concerning dense metric subspaces of certain non-metric spaces. Fund. Math. **48** (1959), 15–25.

If Σ is a Moore space satisfying Axioms 0 and 1 then Σ contains a subspace Σ' which is dense in Σ and is complete metric. A sufficient condition is given on a subset M of Σ in order that it be such a subspace as Σ'.

M. E. Shanks (Lafayette, Ind.)

1879:

Ungar, Peter; Wyler, Eva. An example in elementary topology. Comm. Pure Appl. Math. **13** (1960), 55.

In addition to the existing examples of bijective continuous maps of a space onto a space which are not homeomorphisms, the authors construct in this note another elementary example.

S. T. Hu (Los Angeles, Calif.)

1880:

Bauer, Heinz. Verallgemeinerung eines Faktorisierungssatzes von G. T. Whyburn. Arch. Math. **10** (1959), 373–378.

The monotone-light factorization theorem, proven originally by Eilenberg and the reviewer (independently) for mappings on a compact metric space was generalized by Valnšteĭn [Dokl. Akad. Nauk SSSR **57** (1947), 319–321; MR **9**, 153] and the reviewer to compact mappings on metric spaces. Also this result was established by the reviewer for mappings on locally compact separable metric spaces such that components of point inverses are compact. Various uniqueness theorems for factorizations and invariance results for local compactness and other properties under closed mappings were established earlier by Valnšteĭn and by the reviewer. In the paper under review these theorems are extended to the case of mappings on non-metric spaces which are locally compact (compactness in the, now usual, covering sense). Also, using some results of Stein, similar theorems are established for holomorphic mappings on complex number spaces and for mappings which are "conservative" in a sense previously used by the author.

G. T. Whyburn (Charlottesville, Va.)

1881:

Whyburn, G. T. Compactness of certain mappings. Amer. J. Math. **81** (1959), 306–314.

Soient X et Y deux espaces topologiques et $f: X \to Y$. f est compacte si et seulement si pour tout compact $K \subset Y$, $f^{-1}(K)$ est compact dans X. L'étude de la compacité de certaines applications est faite ici dans le cas où f est continue et X et Y sont métriques séparables localement compacts.

Soit $Y' \subset Y$; tout $X' \subset X$ tel que $f(X') = Y'$ est appelé une trace de Y'. Pour une application f monotone (pour tout $y \in Y$, $f^{-1}(y)$ est compact connexe), si tout compact $Y' \subset Y$ a une trace compacte (propriété de la trace compacte), tout compact $Y' \subset Y$ a une image inverse compacte. Certaines propriétés rencontrées ailleurs définissent des applications ayant la propriété de la trace compacte. L'auteur donne ensuite certaines conditions pour que la compacité d'une application soit une conséquence d'hypothèses sur la compacité des traces ou des images inverses des frontières de certains voisinages dans Y. Entre autres résultats: Si f est monotone, $f(X) = Y$, Y possédant la propriété de Brouwer, et homéomorphe à l'espace quotient de X défini par f, alors f est compact; d'où l'on peut déduire: toute application monotone d'un plan sur un plan est compacte. *L. Fourès* (Marseille)

1882:

Bessaga, C. On the converse of the Banach "fixed-point principle". Colloq. Math. **7** (1959), 41–43.

Let U be a mapping of an abstract set X into itself such

that each iteration U^n $(n=1, 2, \cdots)$ of U has a unique fixed point, and let K be any real number with $0<K<1$. Then there exists a complete metric ρ for X such that $\rho(U(x), U(x'))\leq K\rho(x, x')$, for all $x, x'\in X$. The construction of ρ requires recourse to a special case of the axiom of choice. *S. Mardešić* (Zagreb)

1883:

Michael, E. Dense families of continuous selections. Fund. Math. **47** (1959), 173–178.

Let X be a metric space, Y be a Banach space, and let $\mathscr{C}(Y)$ denote the family of nonempty closed convex subsets of Y. Let ϕ be a mapping of X into $\mathscr{C}(Y)$ such that $\{x\in X: \phi(x)\cap U\neq\emptyset\}$ is open for every open subset U of Y. The author showed in earlier work [Ann. of Math. (2) **63** (1956), 361–382; MR **17**, 990] that such a ϕ admits a selection; i.e., a continuous mapping f of X into Y such that $f(x)\in\phi(x)$ for every $x\in X$. In this note, he shows that for every infinite cardinal α there is a family Φ of selections for ϕ of cardinality no larger than α such that whenever $x\in X$ and $\phi(x)$ has a dense subset of cardinality no larger than α, then $\{f(x): f\in\Phi\}$ is dense in $\phi(x)$. Two corollaries and a generalization of this theorem are also given.

M. Henriksen (Detroit, Mich.)

1884:

Vopěnka, Petr. Remark on the dimension of metric spaces. Czechloslovak Math. J. **9** (**84**) (1959), 519–522. (Russian. English summary)

The English summary states that the author has proven the following theorem. Let R be a metric space. Then $\dim R\leq n$ if and only if there exists a sequence $\{\alpha_k\}_{k=1}^{\infty}$ of open coverings of R such that mesh $\alpha_k\to 0$ for $k\to\infty$, α_{k+1} refines α_k, and every α_k is of order $\leq n$. This is an improvement of a result of Dowker and Hurewicz [Fund. Math. **43** (1956), 83–88; MR **18**, 56] who showed that $\dim R\leq n$ if and only if there is a sequence $\{\alpha_k\}$ of locally finite open coverings of R such that mesh $\alpha_k\to 0$, the closure of each member of α_{k+1} is contained in some member of α_k, and every α_k is of order $\leq n$.

Haskell Cohen (Baton Rouge, La.)

1885:

Sklyarenko, E. Some remarks on spaces having an infinite number of dimensions. Dokl. Akad. Nauk SSSR **126** (1959), 1203–1206. (Russian)

In connection with a problem posed by P. Aleksandrov on the classification of infinite-dimensional compacta, J. Nagata [Proc. Japan Acad. **34** (1958), 146–149; MR **20** #6088] proved that a compact metric R without isolated points is countable-dimensional (the union of a countable number of 0-dimensional spaces) if and only if there is a continuous map f of the Cantor set D^{∞} onto R such that $f^{-1}(x)$ is finite for every $x\in R$. [For the finite-dimensional case see P. S. Aleksandrov, *Combinatorial topology*, Vol. 1, Graylock Press, Rochester, N.Y., 1956; MR **17**, 882; p. 196, Theorem 4.27.] The author strengthens this result by proving: (1) If a compact metric R is not countable-dimensional (or, equivalently, does not have transfinite dimension) and f is a continuous map of D^{∞} onto R, then there is an $x_0\in R$ such that $f^{-1}(x_0)$ has power c. (2) If R is compact metric, has no isolated points and is countable-dimensional, and F is the set of continuous maps of D^{∞} onto R with the property that $f^{-1}(x)$ is finite for all $x\in R$, then F is dense in the set of all continuous maps of D^{∞} onto R.

The author also gives a characterization of regular second countable spaces which are countable unions of closed finite-dimensional subsets in terms of sequences of finite open coverings. This characterization does not extend to countable-dimensional spaces, which were similarly characterized in terms of sequences of closed coverings by Nagata [see the paper referred to above].

H. Komm (Troy, N.Y.)

ALGEBRAIC TOPOLOGY

See also 1608.

1886:

Wu, Wen-Tsün. Topologie combinatoire et invariants combinatoires. Colloq. Math. **7** (1959), 1–8.

A survey article, which may be summed up by the following excerpt. "La topologie combinatoire, qui concentrait sur elle l'attention des topologues pendant la première trentaine d'années de ce siècle, semble cesser peu à peu de susciter leur intérêt, malgré que tous les invariants topologiques des polyèdres finis se trouvent parmi les invariants combinatoires des complexes abstraits. Il est donc peut-être désirable de poursuivre les recherches dans ce domaine. Ce qui vient d'être exposé ici est destiné a montrer qu'il y a une possibilité de faire avancer ces recherches d'une façon méthodique, en particulier dans l'ordre d'idées des théorèmes établis par W. Mayer [Ann. of Math. (2) **43** (1942), 594–605; MR **3**, 318] et K. Lee [Sci. Record (N.S.) **1** (1957), 279–281; MR **20** #6089] pour des invariants numériques particuliers."

1887:

Wylie, S. Intercept-finite cell complexes. Algebraic geometry and topology. A symposium in honor of S. Lefschetz, pp. 389–399. Princeton University Press, Princeton, N.J., 1957. $7.50.

Let $x<y$ express the relation among cells of an abstract cell complex that x is a face of y. The author defines an "intercept-finite" complex to be a cell complex with the property that for any cells x and y there is only a finite number of cells z with $x<z<y$. This self-dual concept is implied by either of the dual concepts "closure-finite" and "star-finite". On an intercept-finite complex two homology theories are defined intermediate between those suggested by finite and arbitrary chains. Difficulties that arise in providing a continuous pairing between cochains and chains when the coefficients are topological groups make it necessary to define a congregation—an abelian group A furnished with a family of sequences of elements of A which play the role of sequences summing to zero in a topological group. Within the realm of categories a duality theory is then developed. *S. Stein* (Zbl **78**, 154)

1888:

Dedecker, Paul. Sur la cohomologie non abélienne. I. Canad. J. Math. **12** (1960), 231–251.

In a previous work of the author [Acad. Roy. Belg. Bull. Cl. Sci. (5) **41** (1955), 1132–1146; MR **19**, 973], 2-

dimensional cohomology of a space with coefficients in a sheaf of non-abelian groups was introduced; but there were a certain number of difficulties. In the present paper, the author has overcome these difficulties by the aid of groupoids. *S. T. Hu* (Los Angeles, Calif.)

1889:

Novikov, S. P. Cohomology of the Steenrod algebra. Dokl. Akad. Nauk SSSR **128** (1959), 893–895. (Russian)

Let $H^*(A)=\sum H^{s,t}(A)$ be the cohomology of the Steenrod algebra A with respect to a given prime p. Adams [Comment. Math. Helv. **32** (1958), 180–214; MR **20** #2711] gave a basis for $H^1(A)$ when $p=2$ and a set of multiplicative relations satisfied by the basis elements; these relations indeed generate all relations of degree ≤ 3. The author obtains further relations, which include a complete set for degrees ≤ 5, and obtains further information about $H^s(A)$ for $s\leq 5$. The methods are based on those of Adams and analogous considerations lead to information about $H^s(A)$ for $p\neq 2$.

The author derives the stable q-stem for $q\leq 14$; besides this, a particularly interesting application of his results is the theorem that, for any r, there exist $\alpha\in\pi_q(S^3)$, $\beta\in\pi_{m+q}(S^q)$, $q>3$, $m>0$, such that, in the stable homotopy ring, $\beta^r\alpha\neq 0$. *P. J. Hilton* (Birmingham)

1890:

Halpern, Edward. On the structure of hyperalgebras. Class 1 Hopf algebras. Portugal. Math. **17** (1958), 127–147.

Continuing his previous work [Mem. Amer. Math. Soc. No. 29 (1958); MR **21** #2982], the author presents two results. The first generalizes a classical result of Samelson, as follows: If H is a Hopf algebra over a field of characteristic 0, which has an anti-commutative and associative coproduct, then H is primitive. This result is false when the field has characteristic $p\neq 0$. The second result uses the notion of class: Let H be an augmented hyperalgebra with coproduct Δ. Let $\pi_0=\pi$, $\pi_\infty=H$, where π is the primitive subalgebra. If $0<k<\infty$, let π_k be multiplicatively generated by the $x\in H$ such that $\Delta'(x)\in\pi_{k-1}\otimes\pi_{k-1}$ (Δ' is, loosely, Δ minus the terms $1\otimes x$, $x\otimes 1$). The class of H is the least k such that $\pi_k=H$. For Hopf algebras of class 1, over fields of characteristic 0, the author determines the coproduct structure, and the product structure of the dual Pontrjagin algebra. *D. W. Kahn* (New Haven, Conn.)

1891:

Browder, William. The cohomology of covering spaces of H-spaces. Bull. Amer. Math. Soc. **65** (1959), 140–141.

Let X be a reasonable H-space, $\pi:\overline{X}\to X$ be a covering space of X, given an H-space structure consistent with that of X. Cartan and Serre have described a spectral sequence for this situation as follows: X is replaced by an equivalent space, fibered over $K(G,1)$, where G is the group of the covering space. The fiber is $\overline{X}$, and the inclusion $i:\overline{X}\to X$ is homotopic to π. The projection is multiplicative.

Using the multiplicative projection map, $f:X\to K(G,1)$, the author turns the spectral sequence into a spectral sequence of commutative Hopf algebras. He then calculates $H^*(\overline{X};Z_p)$ as follows: Let p be an odd prime. $H^*(\overline{X};Z_p)=A\otimes E$, where $A=H^*(X;Z_p)/I$, I is the ideal generated by $f^*(H^*(G,1;Z_p))$, and E is the exterior algebra on generators $x_1,\cdots,x_n$. If the generators of $\ker f^*$ are in dimensions $2p^i$, $1\leq i\leq n$, then $\dim x_i=2p^i-1$. If $p=2$, the result is established for additive structure. *D. W. Kahn* (New Haven, Conn.)

1892:

Okamoto, Sigeru. An algebraic proof of a theorem of I. M. James concerning reduced product complexes. Sci. Rep. Tokyo Kyoiku Daigaku Sect. A **6** (1958), 177–180.

In this note the author gives an algebraic construction to parallel the reduced product construction of James [Ann. of Math. (2) **62** (1955), 170–197; MR **17**, 396]. Almost the same thing was done earlier by Copeland [Proc. Amer. Math. Soc. **7** (1956), 528–534; MR **17**, 1232]. *J. C. Moore* (Princeton, N.J.)

1893:

Moore, John C. On the homology of $K(\pi,n)$. Proc. Nat. Acad. Sci. U.S.A. **43** (1957), 409–411.

Some constructions (in the sense of Cartan [same Proc. **40** (1954), 467–471, 704–707; MR **16**, 390]) are described explicitly for the case that the coefficient domain is the ring of p-adic integers. It is asserted that the pth power of a generator of the group $H^{n+1}(Z_p,n;Z)$ is a primitive element of the group $H^{p(n+1)}(Z_p,n;Z)$. *M. M. Postnikov* (RŽMat **1958** #7587)

1894:

Moore, John C. Semi-simplicial complexes and Postnikov systems. Symposium internacional de topología algebraica [International symposium on algebraic topology], pp. 232–247. Universidad Nacional Autónoma de México and UNESCO, Mexico City, 1958. xiv+334 pp.

Outline of the theory of c.s.s. complexes, groups and fiber spaces, including a description of a Postnikov system as a sequence of twisted Cartesian products. Much of this material appears in Séminaire H. Cartan 1954/55 [École Normale Supérieure, Paris, 1955; MR **19**, 438]; some also in #1895 below. *J. A. Zilber* (Providence, R.I.)

1895:

Barratt, M. G.; Gugenheim, V. K. A. M.; Moore, J. C. On semisimplicial fibre-bundles. Amer. J. Math. **81** (1959), 639–657.

Let $p:E\to B$ be a semisimplicial map. Then Kan has described (E,B,p) as a fibre map if, for all $n>0$, and all t, $0\leq t\leq n$, given $e_i\in E_{n-1}$, $0\leq i\leq n$, $i\neq t$, and $b\in B_n$, with $\partial_i e_j=\partial_{j-1}e_i$ $(i<j;\ i,j\neq t)$, $pe_i=\partial_i b$ $(i\neq t)$, there exists $e\in E_n$ with $pe=b$ and $\partial_i e=e_i$ $(i\neq t)$. Further, (E,B,p) will be called minimal if $\partial_t e$ depends only on the e_i and b. The authors call the map (E,B,p) a fibre bundle with fibre Y if p is onto and if for every map $b:\Delta^n\to B$, of a standard simplex into B, the induced map over Δ^n is (strongly) equivalent to the projection $\Delta^n\times Y\to\Delta^n$. If Y is Kan, then (E,B,p) is called a Kan fibre bundle and a Kan fibre bundle is a fibre map. Given a fibre map we may obtain a minimal fibre map by a fibrewise deformation-retraction, and a minimal fibre map with connected base is a Kan fibre bundle; this reveals the generality of semisimplicial fibre bundles.

Define the group complex $A(Y)$ to be the maximal subgroup of Y^Y; then $A(Y)$ has the universal property that any group complex Γ operating effectively on Y is a subgroup complex of $A(Y)$. If $b \in B_n$ and (E, B, p) is a fibre bundle with fibre Y, then we may associate with b an equivalence $\alpha(b)$: $\Delta^n \times Y \to E^b$ over Δ^n, where $E^b \to \Delta^n$ is induced by b from p. A collection of equivalences $\alpha(b)$, $b \in B$, is called an atlas; moreover $\alpha(b)$ determines $\alpha_i(b)$: $\Delta^{n-1} \times Y \to E^{\partial_i b}$ and, if $b \in B_n$, $(\alpha(\partial_i b))^{-1}\alpha_i(b) \in A(Y)_{n-1}$. These last are called the transformation elements associated with the atlas $\{\alpha(b)\}$. An atlas whose transformation elements lie in Γ is called a Γ-atlas, and two Γ-atlases are Γ-equivalent if they agree modulo Γ. A fibre bundle together with a Γ-equivalence class of Γ-atlases is a Γ-bundle. Γ-maps, Γ-equivalence of Γ-bundles are defined in the natural way.

The apparatus for carrying out the standard constructions of fibre bundle theory in the c.s.s. category is now nearly complete; a further useful tool is that of a twisted Cartesian product $B \tilde{\times} Y$. If the twisting function τ is such that $\tau(b, y) = \xi(b)\partial_0 y$ for some function ξ from B to $A(Y)$ then $B \tilde{\times} Y$ is called a regular twisted Cartesian product (RTCP). It turns out that a fibre bundle whose group can be reduced to Γ is (strongly) equivalent to a RTCP with group Γ (i.e., with $\xi(b) \in \Gamma$ for all b). This enables us to take any Γ-bundle in the form of a RTCP with substantial resulting simplification.

Given a principal Γ-bundle (E, B, p), we may define a right-operation of Γ on E. If Γ operates (on the left) on the complex Y the associated Γ-bundle (E^*, B, p^*) with fibre Y is defined in the usual way by collapsing the product $E \times Y$ under the operation of Γ and defining $p^*(e, y) = pe$; and every Γ-bundle is associated to a unique principal Γ-bundle. The expected classification theorems now follow. The $\overline{W}$-construction $\overline{W}(\Gamma)$ and the acyclic W-construction $W(\Gamma)$ yield the universal Γ-bundle $(W(\Gamma), \overline{W}(\Gamma), p)$ and the following theorem holds. Let Y be a complex on which the group complex Γ operates effectively. The assignment to any map $B \to \overline{W}(\Gamma)$ of the bundle with fibre Y associated to the principal Γ-bundle over B induced from $(W(\Gamma), \overline{W}(\Gamma), p)$ is a 1-1 correspondence between homotopy classes of maps $B \to \overline{W}(\Gamma)$ and (strong) Γ-equivalence-classes of Γ-bundles with base B and fibre Y.

P. J. Hilton (Birmingham)

1896a:

Cartan, Henri. Sur la théorie de Kan. Séminaire Henri Cartan; 9e année: 1956/57. Quelques questions de topologie, Exposé no. 1, 19 pp. Secrétariat mathématique, Paris, 1958. 73 pp. (mimeographed)

1896b:

Cartan, Henri. Sur le foncteur Hom (X, Y) en théorie simpliciale. Séminaire Henri Cartan; 9e année: 1956/57. Quelques questions de topologie, Exposé no 3, 12 pp. Secrétariat mathématique, Paris, 1958. 73 pp. (mimeographed)

1896c:

Cartan, Henri. Théorie des fibrés principaux. Séminaire Henri Cartan; 9e année: 1956/57. Quelques questions de topologie, Exposé no. 4, 12 pp. Secrétariat mathématique, Paris, 1958. 73 pp. (mimeographed)

Exposés 1, 3, and 4 of the Séminaire H. Cartan, École Normale Supérieure, 1956/1957, treat part of the theory of semi-simplicial complexes, Kan complexes, group complexes, and principal fibrations.

J. C. Moore (Princeton, N.J.)

1897:

Kan, Daniel M. A combinatorial definition of homotopy groups. Ann. of Math. (2) **67** (1958), 282–312.

Details of the results announced in Proc. Nat. Acad. Sci. U.S.A. **42** (1956), 542–544 [MR **19**, 440].

1898:

Kan, Daniel M. On homotopy theory and c.s.s. groups. Ann. of Math. (2) **68** (1958), 38–53.

If A and B are topological groups, two homomorphisms $f_0, f_1 : A \to B$ are called loop homotopic if there is a homotopy f_t between them such that each f_t, $0 \leq t \leq 1$, is a homomorphism. Homotopy need not imply loop homotopy. For c.s.s. groups and homomorphisms, the analogous definition requires existence of a c.s.s. homotopy f_I: $I \times A \to B$ such that $f(\alpha, \sigma \cdot \tau) = f(\alpha, \sigma) \cdot f(\alpha, \tau)$ for $\alpha \in I$ and $\sigma, \tau \in A$ (I is the c.s.s. complex of a 1-simplex); loop homotopy need not be an equivalence relation on c.s.s. homomorphisms, although of course it is in the topological case. A c.s.s. group A is called free (with distinguished basis) if (a) for every n, A_n is freely generated, and (b) for every generator $\sigma \in A_n$ and degeneracy operator η^i on σ, $\sigma\eta^i$ is a generator of A_{n+1}. For c.s.s. groups, free groups play a role similar to that played for c.s.s. complexes by Kan complexes (=complexes with extension condition); for example, if A is free, loop homotopy is an equivalence relation for c.s.s. homomorphisms $A \to B$. On the category of reduced complexes (K is reduced if it has only one 0-simplex) the author has defined [#1897] a functor G to the category of free c.s.s. groups; conversely, on the category of c.s.s. groups is defined a functor $\overline{W}$ (essentially the Eilenberg-MacLane construction) to the category of reduced Kan complexes. Restricting G to reduced Kan complexes and $\overline{W}$ to free c.s.s. groups, the author shows that they induce an equivalence between homotopy theory (relative to the base point) in the one category and loop homotopy theory in the other. An example is given of two free c.s.s. groups of the same homotopy type but not the same loop homotopy type.

J. A. Zilber (Providence, R.I.)

1899:

Kan, Daniel M. An axiomatization of the homotopy groups. Illinois J. Math. **2** (1958), 548–566.

In a previous paper [#1897 above], the author defined homotopy groups of c.s.s. (complete semi-simplicial) complexes. In the present paper an axiomatic characterization of homotopy groups on c.s.s. complexes is given which is essentially different from the ones given by Serre, Milnor, Kuranishi, Hu, and others. A theory of homotopy groups on a subcategory S of the category of all c.s.s. complexes with base point is a collection of functors π_n, $n > 0$, from S to the category of groups, together with a natural boundary homomorphism for each fiber space $F \to E \to B$. The axioms are: (1) Weak homotopy equivalences induce isomorphisms on homotopy groups; (2) exactness of the homotopy group sequence of a fiber space; (3) π_1 of a wedge product of two complexes is the free product of π_1 of each of the complexes; (4) $\pi_n(K) \neq 1$ for some K and n.

Uniqueness is proved on the category of c.s.s. complexes weakly homotopy equivalent to countable c.s.s. complexes. Uniqueness is also proved for the category of all c.s.s. complexes when axiom (3) is strengthened to include infinite wedge products and the theory is extended to include π_0. Finally it is shown that axioms (1) and (2) imply that $\pi_1(K(Z, 1))$ is torsion-free and abelian and hence that an arbitrary coefficient group cannot be introduced into the theory.

A somewhat unfortunate feature of this axiomatization is that it requires for its statement, namely axiom (1), the definition of the usual homotopy groups on a c.s.s. complex. One wonders if axiom (1) could not be replaced by a homotopy axiom. *E. H. Brown* (Waltham, Mass.)

1900:

Kan, Daniel M. On monoids and their dual. Bol. Soc. Mat. Mexicana (2) **3** (1958), 52–61.

Let C be a category with products. A monoid structure on an object $A \in C$ is defined to be a map m of $A \times A$ into A satisfying appropriate axioms. A co-monoid structure is defined by reversing all the arrows in the above. It is shown that in the category of groups, a group is free if and only if it admits a co-monoid structure and a group is abelian if and only if it admits a monoid structure. A simplified proof is given for the lemma in #1899 above which states, roughly, that a functor from countable groups to groups which preserves short exact sequences and finite free products is either trivial or equivalent to the inclusion functor. Finally, it is shown that the F construction of Milnor [mimeographed notes, Princeton Univ., 1956] is an isomorphism of the category of c.s.s. complexes onto the category of c.s.s. groups with co-monoid structure. *E. H. Brown* (Waltham, Mass.)

1901:

Kan, Daniel M. A relation between CW-complexes and free c.s.s. groups. Amer. J. Math. **81** (1959), 512–528.

One of the main reasons for the convenience of CW complexes is that when dealing with comparatively simple spaces one can use a decomposition of the space into a CW complex having few cells. The study of the homotopy structure of the space is then simplified. In earlier papers [#1898, #1899 above], the author has shown how to define combinatorially the homotopy groups of c.s.s. complexes, and that if one associates an appropriate c.s.s. complex with a space, such as, for example, its singular complex, then one can obtain the homotopy group of the space. In this scheme one associates with a c.s.s. complex a free c.s.s. group which is in an approximate sense the loop space in the space corresponding to the original complex, and then he defines the homotopy group of the complex to be those of the associated c.s.s. group. In general the resulting c.s.s. group is large and difficult to handle from the point of view of explicit calculation.

In this paper the author shows that one may associate a free c.s.s. group with a CW complex, and that with slight restrictions a CW basis for the c.s.s. group corresponds exactly to the cells of the original CW complex. Thus he obtains a c.s.s. group which will determine the homotopy structure of the space of the CW complex which is smaller than the c.s.s. group determined by earlier procedures, and is in a sense as simple among c.s.s. groups as the original complex was among CW complexes. In the course of carrying out this procedure he shows several analogies between the behavior of free c.s.s. groups and the behavior of CW complexes. The techniques employed and the results of the author are interesting, but it remains to be seen if this procedure will give much information about the homotopy type of spaces.

J. C. Moore (Princeton, N.J.)

1902:

Inoue, Yoshiro. Homotopy groups of css complexes. Math. Japon. **5** (1958/59), 1–16.

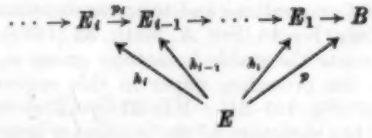
For c.s.s. complexes there exist, essentially, two ways of defining homotopy groups: One, due to A. Heller, which is in effect an axiomatization of obstruction theory [Trans. Amer. Math. Soc. **80** (1955), 299–344; MR **17**, 773]; the second, due to D. Kan [#1897 above], follows the classical geometric theory more closely, and uses the "extension condition"; it is perfectly general, while Heller's theory only deals with the case of "trivial operation" of the fundamental group. The present paper first generalizes the Heller theory to the general case, and then proves the two theories consistent; in particular, it is proved that a c.s.s. complex satisfies the generalized Heller axioms if and only if it satisfies the extension condition.

V. Gugenheim (Baltimore, Md.)

1903:

Hermann, Robert. Secondary obstructions for fibre spaces. Bull. Amer. Math. Soc. **65** (1959), 5–8.

The author describes a method for computing secondary obstructions to cross-sections in fiber spaces. The method is based on the Moore-Postnikov decomposition of a fiber map $F \xrightarrow{i} E \xrightarrow{p} B$. Let F have non-zero homotopy groups in dimensions $n(1) < n(2) < \cdots$ and B be simply-connected. A Moore-Postnikov decomposition of p is a commutative diagram



where p_i is a fiber map with $K(A_i, n(i))$ as fiber and is determined by the invariant $k^{i-1}(E) \in H^{n(i)+1}(E_{i-1}, A_i)$.

Let $f: B_{n(j)} \to E$ be a cross-section of $E \xrightarrow{p} B$ defined over the $n(j)$-skeleton of the CW complex B. The obstruction $w(f)$ to extending f has been defined by W. D. Barcus [Quart. J. Math. Oxford Ser. (2) **5** (1954), 150–160; MR **16**, 160]; $w(f) \in H^{n(j)+1}(B; A_j)$. The map $h_{j-1}f: B_{n(j)} \to E_{j-1}$ can be extended to a cross-section $f': B \to E_{j-1}$, and the first theorem states that $w(f) = f'^{*}k^{j-1}(E)$.

The above result is of particular usefulness in the second obstruction case. If a cross-section $B_{n(1)} \to E$ exists, E_1 may be taken as $B \times K(A_1, n(1))$ and the next invariant computed in some cases by the following theorem: If $A_2 \approx Z$ or Z_p, $k^1(E)$ is characterized by (1) $h_1{}^{*}k^1(E) = 0$, (2) $k^1(E)$ restricted to $K(A_1, n(1))$ is the Eilenberg-MacLane invariant of F, provided it is not zero.

As an illustration of the theory, a simple proof of Liao's formula ($F = S_{n(1)}$, $n(1) > 2$) [Ann. of Math. (2) **60** (1954), 146–191; MR **15**, 979] is given.

J.-P. Meyer (Baltimore, Md.)

1904:

Shiraiwa, Kenichi. Relation between higher obstructions and Postnikov invariants. Nagoya Math. J. **16** (1960), 21–33.

Let $f: K^n \to X$ be a map of the n-skeleton of a CW-complex K into a space X, where X is $(n-1)$-connected and is a space of loops. Assume f can be extended to K^{q+1}. Let $Z_n^{q+2}(f)$ be the set of cohomology classes obtained from obstructions to extending $f': K^{q+1} \to X$, where f' extends f. Let $i: K^n \to K$ be the inclusion, and let $b^n(X) \in H^n(X; \pi_n(X))$ be the fundamental class. In this paper the author shows that $Z_n^{q+2}(f)$ is the image of $i^{q-1} \circ f^*(b^n(X))$ under the higher order cohomology operation defined by the $(q+2)$nd k-invariant of X [as in F. Peterson, Trans. Amer. Math. Soc. **86** (1957), 197–211; MR **21** #4417]. Vaguely stated, the author's theorem states that obstructions defined by filtering on the skeletons of K are the same as those defined by filtering on the Postnikov system of X. The author proves a similar statement about difference cocycles.

F. P. Peterson (Oxford)

1905:

Zisman, Michel. Algèbre caractéristique des espaces fibrés et complexes semi-simpliciaux complets. C. R. Acad. Sci. Paris **247** (1958), 261–263.

A. Borel [Ann. of Math. (2) **57** (1953), 115–207; MR **14**, 490] introduced, given a compact Lie group G and subgroup U, a homomorphism $\rho^*: H^*(B_G) \to H^*(B_U)$, B_G, B_U being the classifying spaces. The author, using the method of twisted cartesian products, replacing each space by its singular complex, gives a semisimplicial construction of a c.s.s. map $\rho: B_U \to B_G$ which induces ρ^*. He applies this method to recover the relationship between the image of ρ^* and the Weyl group of a compact Lie group.

V. Gugenheim (Baltimore, Md.)

1906:

Toda, Hirosi. p-primary components of homotopy groups. IV. Compositions and toric constructions. Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. **32** (1959), 297–332.

Let G_k denote the stable homotopy group $\pi_{n+k}(S^n)$ for n large. In the preceding paper in this sequence [same Mem. **31** (1958), 191–210; MR **21** #4420c] the author determined the structure of the p-primary component of the group G_k for $k \leq 2p^2(p-1)-3$ (p an odd prime). In the present paper he gives generators for all of these p-primary components. These generators are expressed by means of the composition operation and toric construction applied to certain selected elements. The toric construction was introduced by the author in an earlier paper [J. Inst. Polytech. Osaka City Univ. Ser. A. Math. **3** (1952), 43–82; MR **15**, 732]. The precise statements of the results are too complicated to give here.

W. S. Massey (New Haven, Conn.)

1907:

Švarc, A. S. On the genus of a fibred space. Dokl. Akad. Nauk SSSR **126** (1959), 719–722. (Russian)

This paper is a sequel to an earlier one of a similar title [Švarc, same Dokl. **119** (1958), 219–222; MR **21** #1598]. The author presents a construction which can be defined more simply and more generally as follows. Let $p: E \to B$ and $p': E' \to B$ be fibrings. We define a subspace $E+E'$ of the join of E with E'. In $E \times I \times E'$, let $A = \{(x, t, y) \mid x \in E, t \in I, y \in E', p(x) = p'(y)\}$. Form $E+E'$ from $A \cup E \cup E'$ by identifying $(x, 0, y)$ with $x \in E$ and $(x, 1, y)$ with $y \in E'$. Let $p+p': E+E' \to B$ be the map induced by $(x, t, y) \to p(x) = p'(y)$; $p+p'$ is a fibring. When p and p' are sphere bundles, $p+p'$ is the Whitney sum. The author considers the n-fold iteration of this sum with $p = p'$. His techniques for proving his theorem 2′ could in fact yield theorem 2″: Let p and p' be fibrings with the fibres F and F' being $(s-1)$- and $(s'-1)$-connected respectively. If

$$\xi \in H^{s+1}(B, \pi_s(F)) \quad \text{and} \quad \eta \in H^{s'+1}(B, \pi_{s'}(F'))$$

are the first obstructions to cross sections of p and p', then $\xi \cup \eta \in H^{s+s'+2}(B, \pi_{s+s'+1}(F * F'))$ is the first obstruction to a cross section of $p+p'$.

The author relates this construction to the genus of a fibring by theorem 1: Let (E_n, B, F_n, p_n) denote the n-fold sum of (E, B, F, p); then $g(E, B, F, p) \leqq n$ if and only if (E_n, B, F_n, p_n) has a cross section. Theorem 1′: $g(E_n, B, F_n, p_n) = [(g(E, B, F, p)+n-1)/n]$. (Here we assume the author is using [ ] to denote "the integral part of".) The author defines the length "long (E, B, F, p)" of a fibring as the maximum n for which there exist n classes $\xi_i \in H^*(B)$ such that $p^*\xi_i = 0$ and $\xi_1 \cup \cdots \cup \xi_n \neq 0$. [In MR **21** #1598, theorem 4 should have "$x_1 x_2 \cdots x_n \neq 0$" instead of "$x_1, x_2, \cdots, x_n \neq 0$".] The author also defines a cohomology genus of a principal fibring and deduces various relations among the genus, the cohomology genus and the length.

J. Stasheff (Cambridge, Mass.)

1908:

Koseki, Ken'iti. Bemerkung zu meiner Arbeit "Poincarésche Vermutung". Math. J. Okayama Univ. **9** (1959/60), 165–172.

This is a very concrete exposition of the main idea underlying the author's previous proof of the Poincaré conjecture [same J. **8** (1958), 1–106; MR **20** #7270]; unfortunately the author's arguments are logically incorrect.

H. Terasaka (Osaka)

1909:

Hirsch, Morris W.; Smale, Stephen. On involutions of the 3-sphere. Amer. J. Math. **81** (1959), 893–900.

The main theorem asserts that every involution $S^3 \to S^3$ with exactly two fixed points is equivalent to the linear involution $(x_1, x_2, x_3, x_4) \to (x_1, -x_2, -x_3, -x_4)$. One by-product of the methods used is the linear character of an involution of a 3-manifold in the neighborhood of an isolated fixed point. Another is the theorem that the projective plane can be imbedded piecewise linearly in any non-orientable 3-manifold whose fundamental group contains an element of order two which reverses orientation.

[Added in proof: The authors have informed the reviewer that an error has been found in the proof of the main result. The possibility of overcoming this difficulty is viewed with controlled optimism.]

P. A. Smith (New York)

1910:

Wu, Wen-tsün. On the isotopy of a finite complex in a Euclidean space. I, II. Sci. Record (N.S.) **3** (1959), 342–351.

For a topological imbedding $f: X \to R^N$ of a topological space into oriented Euclidean space, the author defines an invariant $\nu_N \in \Theta_f^{N-1}(X)$, an element of the special cohomology group (of P. A. Smith) $\nu_N \in H^{N-1}(\tilde{X}^*)$ where $\tilde{X}^*$ is the topological product $X \times X$ with its diagonal removed;

$t:\bar{X}^*\to\bar{X}^*$ is defined by $t(x,y)=(y,x)$ and $\rho_N=1+(-1)^{N-1}t$. Two topological imbeddings f, g of a space X in R^N are said to be isotopic if there exists a homotopy $F_t: X\to R^N$ such that $F_0=f$, $F_1=g$ and each F_t is an imbedding. Then ${}^{\rho_N}\Theta_f^{N-1}$ is an isotopy invariant of f. A linear imbedding $f:K\to R^N$ of a complex K into R^N is an imbedding which is linear on each simplex. Two linear imbeddings f, g are normal with respect to K if for every pair of disjoint simplices σ, τ of K with $\dim\sigma+\dim\tau\leq N-2$ and any points $x\in|\sigma|$, $y\in|\tau|$, the line joining $f(x)$ and $f(y)$ is not parallel to the line joining $g(x)$ and $g(y)$. For a normal pair of linear imbeddings f and g of K into oriented R^N, an isotopy class ${}^{\rho_N}\Theta_{f,g}^{N-1}(K)\in{}^{\rho_N}H^{N-1}(\bar{K}^*)$ is defined. The author shows there is no generality lost in considering only normal pairs. Then the first paper is ended by proving the relation

$$2\,{}^{\rho_N}\Theta_{f,g}^{N-1}(K)={}^{\rho_N}\Theta_f^{N-1}(K)-{}^{\rho_N}\Theta_g^{N-1}(K).$$

Two linear imbeddings are linearly isotopic if there exists an isotopy between them which is linear on some subdivision of $K\times[0,1]$. Then the main theorem of the second paper says that two normal linear imbeddings f and g of K^n into R^{2n+1} are linearly isotopic if and only if ${}^{1+t}\Theta_{f,g}^{2n}(K)=0$, $n>1$. *S. Smale* (Berkeley, Calif.)

1911:

Hotz, Günter. Ein Satz über Mittellinien. Arch. Math. **10** (1959), 314–320.

Let K be an oriented knot in regular position with respect to a horizontal plane E, and let K be divided up into overpasses and underpasses $A_1, B_1, A_2, B_2, \cdots, A_n, B_n$ in the usual way [Torres and Fox, Ann. of Math. (2) **59** (1954), 211–218; MR **15**, 979]. Let O be the closure of the half-space above E and suppose that $O\cap K=A_1\cup\cdots\cup A_n$. Then, using a base point ∞ that is infinitely high, $\pi(O-K)$ is the free group generated by $\sigma_1,\cdots,\sigma_n$, where σ_i is represented by a loop that winds once around the overpass A_i, $i=1,\cdots,n$. Let t_j be the projection on E of the underpass B_j and let τ_j be the element of $\pi(O-K)$ represented by a path that approximates the path $u_jt_jv_j^{-1}$, when u_j and v_j are paths that descend vertically from ∞ to the initial and final points of t_j respectively. Main Theorem: If there exist integers $\nu_1,\cdots,\nu_n$ such that $\sigma_1^{\nu_1}\tau_1\sigma_2^{\nu_2}\tau_2\cdots\sigma_n^{\nu_n}\tau_n=1$ then K is unknotted.

Obviously, when the integers $\nu_1,\cdots,\nu_n$ are properly chosen $\sigma_1^{\nu_1}\tau_1\sigma_2^{\nu_2}\tau_2\cdots\sigma_n^{\nu_n}\tau_n$ represents a longitude l of the knot, so that the condition of the theorem states that this longitude bounds a 2-cell in $O-K$. The converse of the main theorem is equivalent to the statement that if l bounds in the complement of K it must bound in $O-K$. Whether or not the converse holds is not discussed. If one could prove the converse, one would have an algorithm for deciding whether or not the knot type represented by a given knot diagram is trivial.

R. H. Fox (Princeton, N.J.)

1912:

Hotz, Günter. Arkadenfadendarstellung von Knoten und eine neue Darstellung der Knotengruppe. Abh. Math. Sem. Univ. Hamburg **24** (1960), 132–148.

If K is an oriented knot in regular position relative to E, as in the review above, let $S_1,\cdots,S_n$ denote the projections of the overpasses and $T_1,\cdots,T_n$ the projections of the underpasses. The author calls $S=S_1\cup\cdots\cup S_n$ and $T=T_1\cup\cdots\cup T_n$ partial dissections (Teilzerschneidungen) and S, T a partial dissection pair. He defines certain transformations of a partial dissection (Erweiterung, Reduktion, Isotopie, α-Prozess), analogous to the Reidemeister Ω-transformations of a knot diagram, such that S,T and S',T' define the same knot type if and only if S, T can be transformed by such operations into a partial dissection pair isomorphic to S', T'.

An oriented knot K (in regular position relative to E) is said to be in arcade form (Arkadenfadenlagen) if $K\cap E$ consists of a finite number of points $P_2,\cdots,P_{2n}$, where K pierces E, and a simple arc F from P_{2n+1} to P_1 that contains all double points of the projection. It is proved that every tame knot type contains a knot in arcade form. If two arcade forms belong to the same knot type their projections can be transformed into one another by certain defined arcade-operations.

If K is in arcade form, the injection of $\pi(E-K)$ into the knot group G of K is used to show that G is a homomorphic image of a certain subgroup of the direct product of two free groups. It is alleged that this transforms the word problem for a knot group into the following question: when does an element of the direct product of two free groups lie in a given finitely generated subgroup?

The paper contains several annoying misprints, a crucial function $a_{ST}(w)$ is never really defined (although its meaning is not difficult to guess), and theorem 2 is incorrectly stated (one should read $(\mathfrak{R}_S\cap U(F))\cdot(\mathfrak{R}_T\cap U(F))$ for $\mathfrak{R}_S\cdot\mathfrak{R}_T\cap U(F)$). *R. H. Fox* (Princeton, N.J.)

1913:

Reidemeister, Kurt. Knoten und Geflechte. Nachr. Akad. Wiss. Göttingen. Math.-Phys. Kl. II **1960**, 105–115.

A plat (Geflecht) is the figure formed from a braid on $2n$ strings by joining the initial points $P_1,\cdots,P_{2n}$ in pairs, P_1 with P_2, etc., and also the terminal points $Q_1,\cdots,Q_{2n}$ in pairs, Q_1 with Q_2, etc. Thus a plat is a representative of a knot or link type. The author proves that every tame knot type is represented by at least one plat.

R. H. Fox (Princeton, N.J.)

1914:

Hashizume, Y. Errata: On the uniqueness of the decomposition of a link. Osaka Math. J. **11** (1959), 249.

Article is in same J. **10** (1958), 283–300 [MR **21** #1607].

DIFFERENTIAL TOPOLOGY

See also 1909.

1915:

Reinhart, Bruce L. Line elements on the torus. Amer. J. Math. **81** (1959), 617–631.

An oriented field F of line elements on the torus T^2 may be considered as a map F of T^2 into the sphere S^1; when the field is not oriented, instead of S^1 we have the projective space P^1. If $F:T^2\to S^1$, let $F_*:H_1(T^2)\to H_1(S^1)$ be the induced homology map. One of the main results of the paper is that when $F_*\neq0$ there exists a closed integral curve whose homology class generates the kernel; F_* is then characterized by a pair of integers. The paper contains also a new and simpler proof of the classification theorem of Kneser [Math. Ann. **91** (1924), 135–154] which exhibits qualitatively all possible line elements on T^2.

From it the author derives a necessary and sufficient condition in order that a line element field be orientable.

M. M. Peixoto (Rio de Janeiro)

1916:

Sternberg, Shlomo; Swan, Richard G. On maps with nonnegative Jacobian. Michigan Math. J. **6** (1959), 339–342.

This note contains a proof of the following theorem: Let M and N be two oriented n-dimensional differentiable manifolds with M compact and N connected. Let f be a differentiable map of M into N whose Jacobian $J(f)$ is non-negative. Then either $J(f)=0$, or N is compact, f is onto, and f has positive degree on each component of M on which $J(f)\neq 0$. Some classical results about degrees of maps are incidentally proved in modern terminology without use of triangulations. *S. S. Cairns* (Urbana, Ill.)

1917:

Eells, James, Jr. A generalization of the Gauss-Bonnet theorem. Trans. Amer. Math. Soc. **92** (1959), 142–153.

In ihrer Arbeit [Comment. Math. Helv. **32** (1958), 165–179; MR **21** #868] haben Allendoerfer und der Verf. analog zu den klassischen Sätzen von de Rham den Cohomologiering $H^*(Y, A)$ einer differenzierbaren Mannigfaltigkeit Y, wo A ein Integritätsbereich ist, mit Hilfe von Differentialformen mit Singularitäten beschrieben. Im Rahmen dieser Theorie gibt Verf. nun eine Darstellung für die Stiefel-Whitneyschen Klassen von Y an. Y sei orientierbar und mit einer Riemannschen Metrik versehen. Für jedes r ($1\leq r\leq n$) definiert Verf. die r-te Gauss-Krümmungsform $\Omega^{(r)}$ und die r-te geodätische Krümmungsform $\Phi^{(r)}$ als Formen auf $S_{n-r}(Y)$, dem Totalraum des Bündels der orthonormierten $(n-r)$-Beine von Y. Es sei f_{r-1} ein Schnitt in $S_{n-r+1}(Y)$, der ausserhalb eines in Y gelegenen $(n-r)$-dimensionalen Polyeders definiert ist. f_r sei ein Schnitt in S_{n-r}, der das $(n-r)$-Bein-Feld der $n-r$ ersten Vektoren von f_{r-1} erweitert und ausserhalb eines $(n-r-1)$-dimensionalen Polyeders definiert ist, das Teilpolyeder des vorher erwähnten $(n-r)$-dimensionalen Polyeders ist. Verf. zeigt, dass für r ungerade oder für $r<n$ das Formenpaar $(f_r^*\Omega^{(r)}, f_{r-1}^*\Phi^{(r-1)})$ ein (Z, r)-Paar ist, welches die r-te ganzzahlige Stiefel-Whitneysche Klasse repräsentiert (im Sinne von Allendoerfer-Eells, loc. cit.). Analoges gilt für r gerade mit Z ersetzt durch Z_2. "These integral formulas for the Stiefel-Whitney classes, in a different form and derived by other methods, have beeen discovered by Allendoerfer [Ann. of Math. (2) **51** (1950), 551–570; MR **11**, 689], using forms on the $(n-r)$-skeleton of a particular cellular subdivision (and expressed in polar coordinates defined in each cell of that skeleton). Our formulas should be considered as reinterpreting Allendoerfer's, emphasizing the geometric role played by the forms $\Omega^{(r)}$ and $\Phi^{(r-1)}$." Verf. weist auf die Arbeiten von Chern [Proc. Nat. Acad. Sci. U.S.A. **30** (1944), 269–273; MR **6**, 106], Takizawa [Mem. Coll. Sci. Univ. Kyoto Ser. A. Math. **28** (1953), 1–10; MR **15**, 646] und Aragnol [C. R. Acad. Sci. Paris **238** (1954), 2387–2389; MR **16**, 75] hin.

F. Hirzebruch (Bonn)

1918:

Massey, W. S. On the Stiefel-Whitney classes of a manifold. Amer. J. Math. **82** (1960), 92–102.

Let $w_i\in H^i(M; Z_2)$ be the ith Stiefel-Whitney class of the tangent bundle to a compact, connected, n-dimensional manifold M. Let $\overline{w}_i\in H^i(M; Z_2)$ be the ith dual Stiefel-Whitney class. Many relations between these classes are known. For example, if n is odd, then $w_n=0$, and for any n, $\overline{w}_n=0$. The author extends these results by proving the following theorem. Let M be a compact, n-dimensional manifold and let q be such that $0<q<n$. If $\overline{w}_{n-q}\neq 0$, then there exist integers $h_1, \cdots, h_q$ such that $h_i\geq h_{i+1}\geq 0$ and $n=2^{h_1}+2^{h_2}+\cdots+2^{h_q}$. Furthermore, if M is orientable, then (a) $q\neq 1$, (b) if $n\equiv 2 \pmod 4$, then $h_q\neq 1$, and (c) an odd number of the h_i's are not equal to h_q+1. The proof uses the methods of Wu [C. R. Acad. Sci. Paris **230** (1950), 508–511; MR **12**, 42] and many properties of Steenrod squares. By similar methods, the author also proves the following two theorems and gives examples to show that they are best possible in this direction: if n is even and M is orientable, then $w_{n-1}=0$; and if $n\equiv 3 \pmod 4$ and M is orientable, then $w_n=w_{n-1}=w_{n-2}=0$. {The author notes the following misprints: w_{n-q} should be $\overline{w}_{n-q}$ in the statement of theorem I, and w_{n-1} should be $\overline{w}_{n-1}$ in the footnote at the bottom of page 95.} *F. P. Peterson* (Oxford)

1919:

Liao, S. D. On the imbedding of projective spaces in differentiable manifolds. Sci. Sinica **8** (1959), 781–793.

The author studies the minimum dimension p such that an n-dimensional real projective space P^n can be differentiably imbedded in E^p. H. Hopf proved that P^n cannot be imbedded in E^{n+1} for $n>1$. S. S. Chern proved that P^n cannot be differentiably imbedded in E^{n+2} if $n\neq 2^k-1$, $k\geq 2$ and $n\neq 2^k-2$, $k\geq 2$. The main goal of the author is to take care of one of the two exceptional cases in Chern's work. His theorem is that P^n of even dimension ≥ 4 cannot be differentiably imbedded in a compact orientable differentiable manifold N^{n+2} if $H_1(N^{n+2}; Z_2)$ and $H_2(N^{n+2}; Z_2)$ vanish. To prove the theorem the author studies the homology of a 2-fold covering of the complement of an open tubular neighborhood of P^n in N^{n+2} under the covering transformation.

S. Smale (Berkeley, Calif.)

1920:

Chern, S. S. ★Complex manifolds. Textos de Matemática, No. 5. Instituto de Física e Matemática, Universidade do Recife, 1959. v+181 pp. (polycopiées)

These lecture notes cover much of the recent work on complex manifolds, and are a welcome addition to the literature. Topics covered include: harmonic forms, Kähler manifolds, complex tori, connections, complex line-bundles and sheaves. There are fairly complete proofs of Kodaira's theorem on Hodge manifolds and of Hirzebruch's Riemann-Roch theorem.

M. F. Atiyah (Cambridge, England)

1921:

Weil, André. ★Introduction à l'étude des variétés kählériennes. Publications de l'Institut de Mathématique de l'Université de Nancago, VI. Actualités Sci. Ind. no. 1267. Hermann, Paris, 1958. 175 pp.

"La théorie des variétés kählériennes a pris un grand essor depuis un quart de siècle. Ces variétés furent définies pour la première fois, semble-t-il, dans une note de Kähler de 1932 [Abh. Math. Sem. Univ. Hamburg **9** (1932), 173–186]. Mais leur importance n'apparut qu'à la suite des

premiers travaux de Hodge et surtout de l'exposé d'ensemble qu'il en donna au chapitre V de son livre *The theory and applications of harmonic integrals* [Cambridge University Press, 1941; MR **2**, 296], où, indépendamment de Kähler, il entreprend l'étude systématique de la métrique 'kählérienne' qu'on peut définir sur toute variété algébrique sans point multiple plongée dans un espace projectif, et en tire des conséquences très importantes pour la géométrie algébrique. C'est principalement dans la direction ainsi inaugurée par lui que les recherches se sont poursuivies depuis lors."

Dieses Buch ist aus Vorlesungen entstanden, die der Verf. in Chicago und Göttingen gehalten hat. Verf. sagt, dass das Buch sein Ziel erreicht habe, wenn es die Lektüre der Arbeiten von Kodaira und seiner Schüler und anderer neuerer Arbeiten über kählersche Mannigfaltigkeiten erleichtern würde. Das Buch erfüllt diese Bedingung in sehr ansprechender Weise. Ein Student höherer Semester wird zwar beim Studium wegen der manchmal recht knappen Darstellung und der vielen interessanten Dinge, die zwischen den Zeilen stehen, gewisse Schwierigkeiten überwinden müssen, jedoch benötigt er nicht zu umfangreiche Vorkenntnisse, da Verf. die Verwendung allgemeiner Theorien (Garben, gefaserte Räume) vermeidet und die erforderlichen Sätze (zum Beispiel über Geradenbündel und deren Chernsche Klassen in Kapitel V) direkt formuliert und beweist. Grundlage der Theorie ist natürlich die Existenz der Operatoren H und G, die Verf. nicht beweist. Er kann jedoch dafür auf das Buch von de Rham [*Variétés différentiables*, Paris, Hermann, 1955; MR **16**, 957] verweisen. Die formalen Eigenschaften dieser Operatoren werden vom Verf. zusammengestellt. Das Buch ist eine hervorragende Hilfe für jeden Dozenten, der über kählersche Mannigfaltigkeiten lesen möchte.

In der folgenden Inhaltsangabe wird Ref. aus dem umfangreichen Material jedes Kapitels ziemlich willkürlich wenige Dinge hervorheben.

In Kapitel I, das nur 19 Seiten hat, wird die äussere Algebra über einem komplexen Vektorraum mit hermitescher Metrik studiert. Verf. beginnt mit den elementarsten Dingen, führt den Leser aber bereits bis zu den Operatoren L, $*$, Λ, den primitiven Formen und den entsprechenden Zerlegungssätzen. Das Kapitel enthält sozusagen die punktalen Sätze, die in dem Tangentialraum in einem festen Punkt einer hermiteschen und insbesondere einer kählerschen Mannigfaltigkeit gelten.

Dagegen behandelt Kapitel II die lokale kählersche Geometrie mit den lokalen Operatoren d, d', d'', δ, Δ. Der Operator Δ ist mit L und Λ vertauschbar. Die Zerlegungssätze von Kapitel I führen deshalb zu entsprechenden Sätzen für harmonische Formen. Verf. entwickelt die Theorie vom Standpunkt der integrierbaren fast-komplexen Strukturen aus. Das ist für den Leser besonders instruktiv, obwohl ja kein Unterschied zu den komplexen Strukturen besteht (Newländer-Nirenberg).

In Kapitel III werden kählersche Metriken konstruiert für Untermannigfaltigkeiten, Überlagerungsmannigfaltigkeiten und—in gewissen Fällen—für Quotientenmannigfaltigkeiten kählerscher Mannigfaltigkeiten. Im letzten Fall geht man von einer kählerschen Metrik aus, die bei allen holomorphen Automorphismen invariant ist (Bergmannsche Metrik).

In Kapitel IV werden die fundamentalen Operatoren H und G, die ja keinen lokalen Charakter haben, für kompakte kählersche Mannigfaltigkeiten eingeführt und ihre Relationen mit den bisher definierten Operatoren besprochen. Für eine kompakte kählersche Mannigfaltigkeit V ist der Cohomologie-Modul $H(V)$ die direkte Summe der Moduln $H^{p,q}(V)$. In Zusammenhang mit den Zerlegungssätzen bezüglich primitiver Formen folgt als Anwendung insbesondere der Indexsatz von Hodge. Das Kapitel schliesst mit Sätzen über Hodgesche Mannigfaltigkeiten.

Kapitel V (Fonctions de transition et diviseurs). Satz von Kodaira-Spencer, dass jede ganzzahlige Cohomologieklasse von Typ (1, 1) als Cohomologieklasse eines holomorphen Geradenbündels auftritt. Funktionen mit Multiplikatoren. Theta-Funktionen.

Kapitel VI (Tores complexes, fonctions thêta, variétés abéliennes). Die Theorie der kählerschen Mannigfaltigkeiten wird auf komplexe Tori angewandt. Für diesen Spezialfall wurde die Existenz der Operatoren H und G in Kapitel IV vollständig bewiesen und zwar ohne Anlehnung an den allgemeinen Existenzbeweis von de Rham, der ja im vorliegenden Buch nicht behandelt wird. "Je n'ai pu néanmoins résister à la tentation d'insérer un chapitre traitant de la théorie des fonctions thêta et des variétés abéliennes sur le corps des complexes. Cette théorie peut être considérée comme celle d'un type particulier de structures kählériennes (les structures invariantes sur un tore), et c'est même ce point de vue qui conduit à la démonstration la plus naturelle d'un des principaux théorèmes d'existence de la théorie (le théorème dit 'd'Appell-Humbert')."

Appendice (Propriétés élémentaires des diviseurs sur les variétés complexes). *F. Hirzebruch* (Bonn)

1922a:

Calabi, Eugenio; Vesentini, Edoardo. Sur les variétés complexes compactes localement symétriques. Bull. Soc. Math. France **87** (1959), 311–317.

1922b:

Calabi, Eugenio; Vesentini, Edoardo. On compact, locally symmetric Kähler manifolds. Ann. of Math. (2) **71** (1960), 472–507.

The first of these papers is an announcement of material presented in detail in the second. These papers are concerned chiefly with the study of the cohomology groups of compact Kähler manifolds X which are uniformizable by Cartan domains with coefficients in the sheaf Θ of germs of holomorphic tangent vector fields. The basic approach to the problem is through the relation between non-existence of nontrivial harmonic tensor fields and the vanishing of certain cohomology groups $H^q(X, \Theta)$. The authors confine themselves to the above-mentioned spaces because for these spaces there are especially simple methods of calculating the curvature tensors. The method used in these papers leans heavily on the classification of certain Cartan domains. (Another method of making these calculations is given by A. Borel; see following review.) The main results may be stated as follows.

Theorem 1: For each irreducible Cartan domain D there exists an integer $\gamma(D)$ satisfying $\sqrt{(2n)} < \gamma(D) \leq n+1$ ($n = \dim_C D$), depending only on the curvature of the invariant metric of D, with the following property. For any compact locally symmetric, complex manifold X, uniformizable by (i.e., whose universal covering space is) D, the cohomology groups $H^q(X, \Theta)$ vanish for all q satisfying $0 \leq q < \gamma(D) - 1$. In particular, the only case where $\gamma(D) = 2$

is where $\dim_C D = 1$; that is to say, when D is equivalent to the unit disk.

Theorem 2: The complex analytic structure of any compact, complex manifold X that is uniformizable by an irreducible Cartan domain D of at least two complex dimensions is locally rigid, even if the complex structure of the universal covering space is allowed to vary. In other words, the number of moduli of a complex manifold X with the stated properties is zero.

Theorem 3: Let X be an n-dimensional, compact, complex manifold, whose universal covering space is a Cartan domain D. Then the Θ-characteristic number

$$\chi(X, \Theta) = \sum_{q=0}^{n} (-1)^q \dim_C H^q(X, \Theta),$$

the arithmetic genus (Todd genus)

$$\chi(X, \Omega^0) = \sum_{q=0}^{n} (-1)^q \dim_C H^q(X, \Omega^0)$$

(where Ω^0 is the sheaf of germs of holomorphic functions), and the Euler-Poincaré characteristic

$$\chi(X) = \chi(X, R) = \sum_{q=0}^{2n} (-1)^q \dim_R H^q(X, R)$$

are mutually related by the proportionality formula

$$\chi(X, \Theta) = a_1(D)\chi(X, \Omega^0) = a_2(D)\chi(X) = R^n(X)a_3(D)v(X),$$

where $R(X)$ is the value of the (constant) scalar curvature of X, and $v(X)$ is the volume of X. Moreover, $a_1(D)$ is a positive integer, $a_2(D)$ is a positive rational number, $a_3(D)$ a positive real number and $R(X) < 0$. The same relation, in addition, is valid, with the same proportionality constants $a_j(D)$, if one replaces X with the compact, symmetric space Y corresponding to D, with positive scalar curvature $R(Y)$.

Theorem 4: Let D be a Cartan domain, reducible or not, whose irreducible factors have at least two complex dimensions; let G be the group of all holomorphic (isometric) transformations of D, represented as an algebraic group of real matrices, and let Γ be a discrete subgroup of G acting on D with compact quotient space. Then there is an inner automorphism of G, such that the components of all matrices representing the transformation of Γ generate a simple algebraic extension of the rational number field. *L. Auslander* (New Haven, Conn.)

1923:

Borel, Armand. On the curvature tensor of the Hermitian symmetric manifolds. Ann. of Math. (2) **71** (1960), 508–521.

Author's summary: "This note is a complement to a paper of E. Calabi and E. Vesentini [reviewed above]. In that work the authors attach to an irreducible bounded symmetric domain D a number $\gamma(D)$, depending on the curvature of D with respect to an invariant kählerian metric; this number is related to the vanishing of certain cohomology groups on the compact quotients of D by discrete groups of automorphisms. By definition $\gamma(D) = R/n\lambda_1$, where $n = \dim_C D$, R is the (constant) scalar Riemannian curvature of D, and λ_1 the smallest eigenvalue of a certain linear operator Q, self-adjoint with respect to the metric, defined in terms of the curvature tensor. The trace of Q is equal to $R/2$. This number is computed in the paper reviewed above for the four main classes of bounded symmetric domains, with the help of an explicit representation of the domain. Our purpose here is to discuss it from the point of view of Lie algebras. The computations to which we are thus led can be performed in all cases, but we shall confine ourselves to the two exceptional domains omitted in the paper reviewed above."

L. Auslander (New Haven, Conn.)

1924:

Vesentini, Edoardo. On a class of compact Kähler varieties. I, II. Ann. Mat. Pura Appl. (4) **46** (1958), 183–200 (Italian summary); **47** (1959), 75–79.

The main result of the first paper is as follows. Let V be a compact connected Kähler variety with a holomorphic projection $\pi: V \to B$, where B is a complex projective space, and $0 < \dim B < \dim V$. If π is of maximum rank at each point of V, then the restriction of the Dolbeault groups $H^q(V, \Omega)$ of V to the fibre $\pi^{-1}(P)$, where P is a generic point of B, is injective in all dimensions. In particular, the geometric genus of V, and the dimensions of the vector spaces of holomorphic p-forms of degree $p < \dim V - \dim B$ are all zero. The restriction that π should be of maximum rank everywhere is a necessary one.

In the second paper some generalisations are considered. The main theorem is now the following. Let V and π be as before, and let B be an irreducible non-singular projective variety, with $\dim B < \dim V$. Let $\tilde{F}$ be a complex analytic line-bundle on B, whose real characteristic class contains a real d-closed $(1, 1)$-form whose associated hermitian form is negative semidefinite at each point of B. Then, if π is of maximum rank everywhere, the restriction of $H_0^0(V, \Omega^q(F))$ to the fibre $[F = \pi^* \tilde{F}]$ is injective for all q. Here $H_0^0(V, \Omega^q(F))$ is the maximal subspace of holomorphic $(q, 0)$-forms ω with coefficients in F no one of which is such that its restriction to each fibre is zero.

J. A. Todd (Cambridge, England)

1925:

Kobayashi, Shoshichi. Remarks on complex contact manifolds. Proc. Amer. Math. Soc. **10** (1959), 164–167.

On définit une structure de contact complexe dans une variété complexe M à $2n+1$ dimensions par un recouvrement ouvert $\{U_i\}$ de M et une 1-forme ω_i holomorphe dans chaque U_i satisfaisant aux conditions: (1) il y a une fonction holomorphe f_{ij} dans chaque intersection non vide $U_i \cap U_j$ telle que $\omega_i = f_{ij}\omega_j$; (2) la $(2n+1)$-forme $\omega_i \wedge (d\omega_i)^n$ n'est nulle part nulle dans U_i. Le fibré linéaire holomorphe sur M défini par le système $\{f_{ij}\}$ de fonctions de transition a une classe de cohomologie de degré 2 comme classe caractéristique, désignée par α. Théorème: Soit M une variété complexe à $2n+1$ dimensions complexes, munie d'une structure de contact complexe. On vérifie alors sur M les propriétés suivantes. (1) Le groupe de structure du fibré holomorphe tangent à M peut se réduire au groupe $\mathrm{U}(1) \times (\mathrm{Sp}(n) \otimes \mathrm{U}(1))$. (2) La classe caractéristique (de Chern) du fibré tangent $1 + c_1(M) + \cdots + c_{2n+1}(M)$ se décompose selon le cup-produit $(1+\alpha)(1+n\alpha+\cdots)$; on a en particulier $c_1(M) = (n+1)\alpha$. (3) Le fibré linéaire holomorphe L défini par le système de fonctions de transition $\{f_{ij}^{-1}\}$ est associé (après une réduction de son groupe de structure au sous-groupe compact maximal) à un fibré principal P sur M avec groupe de structure $U(1)$; l'espace total de P admet, d'une façon naturelle, une structure de contact réelle à $4n+3$ dimensions.

Le théorème énoncé ci-dessus est illustré par des exemples connus de variétés complexes avec structure de contact complexe, à savoir, les espaces projectifs complexes à $2n+1$ dimensions et l'espace total du fibré des éléments cotangents projectifs complexes sur une variété complexe quelconque. *E. Calabi* (Minneapolis, Minn.)

1926:

Singer, I. M. The geometric interpretation of a special connection. Pacific J. Math. **9** (1959), 585–590.

Es sei $\mathfrak{B}=(B, M, G)$ ein differenzierbares Prinzipalbündel über der differenzierbaren Mannigfaltigkeit M, wobei vorausgesetzt werde, dass G die Komplexifizierung einer kompakten Lieschen Gruppe K ist. Die differenzierbare Mannigfaltigkeit B sei mit einer fast-complexen Struktur versehen, die invariant ist unter den Rechtstranslationen vermöge Elementen von G und die folgende Eigenschaft hat: Für $c \in B$ führt die Multiplikation mit i (bezüglich der fast-komplexen Struktur) den Tangentialraum an die Faser durch c in sich über. Jede Faser ist also fast-komplex. Diese fast-komplexe Struktur soll mit der zweiseitig invarianten komplexen Struktur von G übereinstimmen.

Zu jeder Reduktion der Strukturgruppe von G auf K gibt Verf. eine geometrische und koordinatenfreie Konstruktion eines Zusammenhanges H in $\mathfrak{B}$ an, für den gilt: (1) H ist invariant unter der fast-komplexen Struktur, d.h. jeder der "horizontalen Teilräume" H_c ($c \in B$) geht bei Multiplikation mit i in sich über. (2) Man bezeichne mit B_K den Totalraum des Prinzipalbündels mit Faser K, der zur Reduktion der Strukturgruppe gehört. B_K ist Untermannigfaltigkeit von B. Für $c \in B_K$ soll H_c im Tangentialraum an B_K in c enthalten sein.—Durch die Reduktion und die Bedingungen (1) und (2) ist der Zusammenhang eindeutig bestimmt. Die Konstruktion des Verf. kann insbesondere auf das komplexe Tangentialbündel einer komplexen Mannigfaltigkeit angewandt werden. Man erhält so zu jeder hermiteschen Metrik einen Zusammenhang. Wenn die hermitesche Metrik kählersch ist, dann geht dieser Zusammenhang durch Restriktion aus dem Riemannschen Zusammenhang der entsprechenden Riemannschen Metrik hervor.

"An esthetic problem remaining in the foundations of Riemannian geometry is this. To each Riemannian metric on a manifold one associates a connection on its bundle of bases. This connection is uniquely characterized by its having zero torsion and its parallel translation preserving the metric. By very definition at each tangent space of the bundle the connection assigns a complement to the vertical. Yet the various existence proofs of the Riemannian connection are computational and do not exhibit this complementary space (the horizontal space) directly. The problem is how can one do so directly from the metric, given (say) by the reduction of the bundle to the bundle of frames."

Dieses "ästhetische Problem" wurde vom Verf. in speziellen Fällen gelöst. *F. Hirzebruch* (Bonn)

1927:

Murakami, Shingo. Sur certains espaces fibrés principaux différentiables et holomorphes. Nagoya Math. J. **15** (1959), 171–199.

X désigne une C-variété au sens de Wang (i.e., une variété analytique complexe, compacte, simplement connexe, dont le groupe des automorphismes holomorphes est transitif); A désigne un groupe de Lie complexe, abélien, connexe, et $\mathbf{C}^m$ le revêtement universel de A. Les classes de fibrés principaux holomorphes [resp. différentiables] de base X, de groupe A, forment un groupe abélien $H^1(X, \mathbf{A}_h)$ [resp. $H^1(X, \mathbf{A}_d)$], et la suite

$$0 \to H^1(X, \mathbf{C}^m_h) \to H^1(X, \mathbf{A}_h) \to H^1(X, \mathbf{A}_d) \to 0$$

est exacte (th. 2). Soit $X=G/U$, G étant un groupe de Lie complexe, semi-simple, simplement connexe, qui opère holomorphiquement dans X de manière "presque effective"; alors l'homomorphisme $\mathrm{Hom}_h(U,A) \to H^1(X,\mathbf{A}_h)$ qui, à chaque homomorphisme holomorphe $U \to A$, associe la classe du fibré holomorphe de base X, de groupe A, "associé" au fibré G (de base X, de groupe U), est bijectif (th. 3). Soit maintenant K un groupe de Lie compact, semi-simple, connexe, qui opère transitivement dans X par des transformations holomorphes; alors tout fibré principal holomorphe P, de base X, de groupe A, admet une connexion de type (1, 0) dont la forme de courbure est de type (1, 1) et invariante par K; de plus, une telle connexion est unique si X est kählérienne (th. 4). Enfin, soit P un fibré principal holomorphe de base X, de groupe A; si P admet une connexion holomorphe, P est trivial (th. 5).

Le début de ce travail concerne le cas plus général où X est la base d'un fibré principal différentiable Y, de groupe B, et où A est un groupe de Lie abélien connexe; soit θ: $\mathrm{Hom}(B, A) \to H^1(X, \mathbf{A}_d)$ l'homomorphisme qui, à chaque homomorphisme différentiable $B \to A$, associe la classe du fibré différentiable de base X, de groupe A, "associé" à Y; si Y est simplement connexe et B compact, la suite $0 \to \mathrm{Hom}(B, A) \xrightarrow{\theta} H^1(X, \mathbf{A}_d) \to H^1(Y, \mathbf{A}_d)$ est exacte (th. 1).

La démonstration du th. 1 se rattache à un article de J. L. Koszul [même J. **15** (1959), 155–169; MR **22** #982]. La démonstration des autres théorèmes fait intervenir la technique des algèbres de Lie et des sous-algèbres de Cartan. *H. Cartan* (Paris)

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